

On dynamics and bifurcations of area-preserving maps with homoclinic tangencies.

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Abstract

We study bifurcations of area-preserving maps, both orientable (symplectic) and non-orientable, with quadratic homoclinic tangencies. We consider one and two parameter general unfoldings and establish results related to the appearance of elliptic periodic orbits. In particular, we find conditions for such maps to have infinitely many generic (KAM-stable) elliptic periodic orbits of all successive periods starting at some number.

1 Introduction.

The present paper is devoted to the study of bifurcations of area-preserving maps (APMs) with quadratic homoclinic tangencies. The case of two-dimensional symplectic (area-preserving and orientable) maps was analyzed in the papers [1, 2, 3, 4, 5]. Closely related bifurcation problems were considered in the papers [6, 7, 8, 9, 10] where bifurcations of conservative flows with a homoclinic loop of a saddle-focus equilibrium were studied. In the works [6, 7] the case of three-dimensional divergence-free flows was considered, while in [8, 9, 10] the dynamical behaviour and bifurcations in two degrees of freedom Hamiltonian systems were analyzed.

In the present paper we do not restrict ourselves to symplectic maps, but we also consider the new case of area-preserving and non-orientable maps. First, we give a classification of APMs with quadratic homoclinic tangencies and, further, prove certain theorems on the existence of infinitely many bifurcations (cascades) leading to the appearance of generic (KAM-stable) elliptic periodic orbits.

We recall that for dissipative systems, the related problems are quite traditional and many results obtained here are of fundamental importance in the theory of dynamical chaos. One of such results, known as *theorem on cascade of periodic sinks*, goes back to the famous papers of Gavrilov and Shilnikov [11] and Newhouse [12], see also [13, 14]. This theorem deals with the so-called sectionally dissipative case, i.e., when a homoclinic tangency is associated to a saddle

fixed (periodic) point with multipliers $\lambda_1, \dots, \lambda_n, \gamma$ such that $|\lambda_i| < 1, |\gamma| > 1$ and the *saddle value* $\sigma \equiv |\gamma| \cdot \max_i |\lambda_i|$ is less than 1. In this case, bifurcations of the homoclinic tangency lead to the appearance of asymptotically stable periodic orbits (periodic sinks). Moreover, in any one parameter general unfolding f_μ , such orbits exist for values of μ forming an infinite sequence (cascade) of intervals that do not intersect and accumulate to $\mu = 0$.

A very nontrivial extension of this (quite simple) result was made by S. Newhouse [15], who proved that, for any such one parameter general unfolding f_μ , there exist intervals in which there are dense values of μ such that the corresponding diffeomorphism f_μ has a homoclinic tangency. Together with the theorem on cascade of periodic sinks, this implies that the values of μ where f_μ possesses *infinitely many periodic sinks* form residual subsets of these intervals, i.e., subsets which are the intersection of a countable number of open and dense sets. Thus, this *Newhouse phenomenon* should be generic for chaotic sectionally dissipative systems allowing homoclinic tangencies. Later, the existence of Newhouse regions (where systems with homoclinic tangencies are dense) was proved for any dimension, [14, 16, 17], as well as for conservative systems, [18, 19], see also [20].

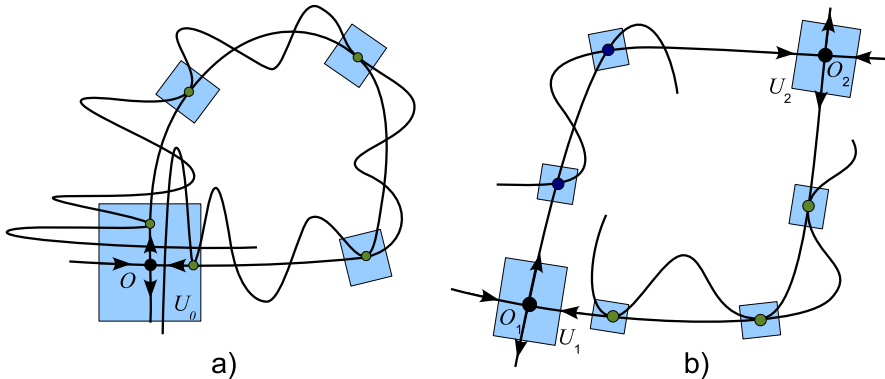


Figure 1: Examples of planar diffeomorphisms with (a) a (quadratic) homoclinic tangency at the points of some homoclinic orbit; (b) a nontransversal heteroclinic cycle containing two saddle fixed points and two selected heteroclinic orbits one of which is nontransversal. Here it is shown also small neighbourhoods of (a) the nontransversal homoclinic orbit and (b) the heteroclinic cycle which look as a union of small rectangles.

In principle, the theorem on cascade of periodic sinks admits various extensions even in the case of two-dimensional diffeomorphisms, see e.g. [21].

Thus, the main bifurcations of quadratic homoclinic tangencies with $\sigma = 1$ were studied in [22, 23] where the so-called *generalized Hénon map*

$$\bar{x} = y, \bar{y} = M_1 - M_2x - y^2 + a_1xy + a_2y^3$$

was derived as the normal form for the first return maps. In this map the parameters M_1 and M_2 are, in fact, the rescaled initial parameters, which control the splitting of the invariant manifolds and the saddle value, respectively. The small coefficients a_1 and a_2 are important: if $a_1 \neq 0$, then the Andronov-Hopf bifurcation of a fixed point (with multipliers $e^{\pm i\psi}$) is non-degenerate; if $a_1 \neq 0$ and $a_2 \neq 0$, then bifurcations at the strong resonant case $\psi = \pi/2$ are non-degenerate, see [24, 25]. Naturally, in the area-preserving case, we have that $|M_2| \equiv 1$, $a_1 \equiv 0$, and a_2 is important again.

Another important extension of the theorem on cascade of periodic sinks concerns the case of two-dimensional diffeomorphisms having nontransversal heteroclinic cycles, i.e., there are several

saddle fixed (periodic) points which form a cycle due to intersections of their invariant manifolds, and some of these intersections are nontransversal (see Figure 1 b). If the saddle values of all these (saddle) points are less than 1 (or greater than 1), then the results look quite similar to the homoclinic case with saddle value $\sigma < 1$ (respectively, $\sigma > 1$), although the intervals of existence of periodic sinks (sources) can intersect here [26]. However, if the heteroclinic cycle contains at least two saddle points such that $\sigma_1 < 1$ and $\sigma_2 > 1$, then, as it was shown in [27], a new dynamical phenomenon called *mixed dynamics* occurs. The essence of this phenomenon consists in the fact that, first, global bifurcations of such systems lead to the appearance of infinitely many coexisting hyperbolic periodic points of all possible types, i.e., saddle, stable and completely unstable (as well as stable and unstable invariant circles [28, 29]); and, second, periodic orbits of one type are not separated from the ones of another type, i.e., the closures of the sets of periodic orbits of different types have nonempty intersections. Note also that the mixed dynamics is a generic phenomenon [27], i.e., it takes place on residual subsets in some open (Newhouse) regions. Especially, this is significant for reversible systems for which the (reversible) mixed dynamics gives rise to the coexistence of *infinitely many saddle, attracting, repelling and elliptic periodic orbits* appearing generically when symmetric homoclinic or heteroclinic structures are involved, see [30, 31]. Note that the phenomenon of reversible mixed dynamics is frequently observed in applications, for example, in a model of coupled rotators [32], in mechanical models such as nonholonomic models of a Celtic stone [33] and a rubber-body [34], etc.

Concerning the corresponding results in the conservative case, we mention, above all, the well-known theorem of S. Newhouse [35] on the emergence of 1-elliptic periodic orbits (with only one pair of multipliers on the unit circle $e^{\pm i\varphi}$ with $\varphi \neq 0, \pi$) under bifurcations of homoclinic tangencies of multidimensional symplectic maps. Note that the Newhouse theorem does not give answer whether these 1-elliptic orbits are generic.¹ However, this fact is very important in the two-dimensional case where an 1-elliptic point is elliptic and the genericity means the KAM-stability of such point. Such a problem was considered in [6, 7] when studying bifurcations of three-dimensional divergence free flows with a homoclinic loop of a saddle-focus equilibrium, and in [1] when studying bifurcations of two-dimensional symplectic maps with quadratic homoclinic tangencies. However, a more or less complete description of related bifurcation diagrams (including the questions about the coexistence of elliptic points of different periods) was not obtained in these papers. This was done in [5] for the symplectic case. In this connection, we note that in [2, 3] it was discovered that APMs with quadratic homoclinic tangencies (at $\mu = 0$) can possess infinitely many coexisting elliptic periodic orbits, and, moreover, these orbits have successive periods $k_0, k_0 + 1, \dots$, starting at some integer k_0 . Thus, such APMs display the *phenomenon of global resonance* leading to strict ordering even in the structure of elliptic points.²

In the present paper the results of [5] and [2, 3] are significantly extended, in particular including into consideration non-orientable APMs with quadratic homoclinic tangencies. Note

¹The birth of 2-elliptic generic periodic orbits was proved in [36, 37] for the case of four-dimensional symplectic maps with homoclinic tangencies to saddle-focus fixed points. Recall that a periodic orbit is 2-elliptic if it has two pairs of multipliers $e^{\pm i\phi}$ and $e^{\pm i\psi}$ with $\phi \neq \psi$ and $\phi, \psi \neq \{0, \pi\}$. The genericity means, in particular, that $\phi, \psi \neq \{\pi/2, 2\pi/3\}, \phi \neq 2\psi, \phi \neq 3\psi$, etc.

²It is interesting to note that maps with infinitely many generic elliptic periodic points are dense in the space of APMs with nontransversal heteroclinic cycles, [38, 39]. Moreover, the conditions for the existence of such orbits are closely related to certain arithmetic properties of some numerical invariants (Ω -moduli), whose set includes even the first Birkhoff coefficients from the normal forms of local maps near saddle points. Note also that in [40] it was proved that in the standard map family there exists a residual set of parameter values for which the map has infinitely many elliptic islands accumulating to a locally maximal hyperbolic set. An analogous result for the so-called cyclicity-one elliptic islands was proved recently in [41].

that such systems can be either planar maps, like the non-orientable conservative Hénon map $\bar{x} = y$, $\bar{y} = M + x - y^2$, or area-preserving diffeomorphisms on non-orientable surfaces.

Our paper is organized as follows.

In *Section 2* we state the problem and give the necessary geometric constructions as well as the general technical results. In particular, we formulate, in form of lemmas, several important results on normal forms of saddle APMs including rather new results (e.g. Lemma 2 on the n -th order normal form) for the non-orientable case. In fact, we extend the well-known analytical Birkhoff-Moser normal form (see formula (2)) to the finite-smooth case.

In *Section 3* we give a classification of APMs with quadratic homoclinic tangencies according to the type of the semi-local dynamics, i.e., the type of the structure of the set N of orbits entirely lying in a small neighbourhood U of the contour $O \cup \Gamma_0$, where O is a saddle fixed point and Γ_0 is a homoclinic orbit at whose points the manifolds $W^u(O)$ and $W^s(O)$ have a quadratic tangency. Note that U is represented as a union of a small disk U_0 containing the point O and a finite number of disks surrounding those homoclinic points of the orbit Γ_0 which do not belong to U_0 , see Figure 1. Thus, U_0 contains infinitely many points of Γ_0 lying in $W_{loc}^s(O)$ and $W_{loc}^u(O)$ and accumulating to O . We divide the APMs with quadratic homoclinic tangencies into three classes. In the first class, the set N has always a trivial structure: $N = \{O; \Gamma_0\}$; in the second class, N is nontrivial and allows always a complete description in terms of the symbolic dynamics, see Section 3.3. In the third class, the structure of N can be both trivial ($N = \{O; \Gamma_0\}$) and nontrivial (N contains nontrivial hyperbolic subsets) depending not only on the geometry of the manifolds $W^u(O)$ and $W^s(O)$ near a point of homoclinic tangency³, as in the case of tangencies in the first and second classes, but also on other invariants of the homoclinic structure. In particular, the structure of N depends on an invariant τ (see formula (23)) whose variation near $\tau = 0$ (without splitting the tangency) implies that the set N changes the structure. See the corresponding propositions in Section 3.4.

The central part of the paper, *Sections 4, 5 and 6*, is devoted to the study of the main bifurcations in parameter families f_ε of APMs which unfold generally the initial homoclinic tangency. First of all, we are interested in bifurcations of the so-called *single-round* periodic orbits, i.e., those which pass only once along the neighbourhood $U(O \cup \Gamma_0)$, see Definition 1. Every point of such an orbit can be considered as a fixed point of the corresponding first return map T_k defined in some domain near a homoclinic point. In this paper we construct these first return maps as certain compositions $T_k = T_1 T_0^k$ of the local map T_0 and the global map T_1 , where k runs along all sufficiently large integer numbers. The local map is, in fact, a conservative saddle map which is defined by orbits of the diffeomorphism f_ε on a small neighbourhood (a disk) $U_0 \subset U$ containing the point O_ε , thus, $T_0 = f_\varepsilon|_{U_0}$. The global map T_1 is a map acting by the orbits of f_ε from a small neighbourhood, say Π^- , of a homoclinic point, M^- , belonging to $W_{loc}^u \cap U_0$, to a small neighbourhood, Π^+ , of another homoclinic point, M^+ , belonging to $W_{loc}^s \cap U_0$. Then one can write $T_1 = f_\varepsilon^q|_{\Pi^-}$, where q is a number such that $M^+ = f_0^q(M^-)$, see Figure 2.

We assume here that the set ε of the governing parameters include always the parameter μ of the splitting between the manifolds $T_1(W_{loc}^u)$ and W_{loc}^s near the homoclinic point M^+ . Then we show, see the Rescaling Lemma 6 of Section 4, that every first return map T_k , for sufficiently large k and small μ , can be written in the unified rescaled form

$$\bar{x} = y + o(\lambda^k), \quad \bar{y} = M - \nu_1 x - y^2 + \nu_2 \lambda^k y^3 + o(\lambda^k), \quad (1)$$

³In the case of quadratic homoclinic tangencies such a geometry is completely determined by the signs of 4 parameters: the two multipliers of the point O and two more parameters c and d that characterize the mutual position and orientation of the curves $W^u(O)$ and $W^s(O)$ near a tangency point, see Section 2.2.

where the rescaled coordinates (x, y) and the parameter M can take values on a ball $\|(x, y, M)\| \leq L_k$, where $L_k \rightarrow \infty$ as $k \rightarrow \infty$; ν_1 is the index equal to $+1$ or -1 depending on the orientability of the map T_k ; ν_2 is some invariant of the homoclinic structure. In fact, the map (1) is a generalized conservative Hénon map whose bifurcations are well known. Therefore, we know the bifurcations that single-round periodic orbits undergo: the list of these bifurcations coincides (up to some small details) with the list of bifurcations of fixed points in the map (1), see Sections 4.2.1 and 4.2.2. However, this does not mean that we have studied completely the homoclinic bifurcations, since we need to construct the *bifurcation diagram*, which includes not only the list of bifurcations of the first return maps T_k , but also shows a disposition of these bifurcations in the parameter space. Since we are interested in the bifurcations leading to the appearance of elliptic periodic orbits, first of all, we need to answer the question “Can elliptic orbits of different periods coexist?”

Our first result on this theme, Theorem 1 from Section 5, shows that in the main case, when $\tau \neq 0$ and the homoclinic tangency takes place for $\mu = 0$, in the family f_μ , the intervals of values of μ corresponding to the existence of single-round elliptic periodic orbits of period $(k + q)$ (or double-round ones of period $2(k + q)$ when the maps T_k are non-orientable) are not crossed for different sufficiently large k . However, the “globally resonant case” $\tau = 0$ is much more interesting. Here, the pointed out intervals can intersect and, moreover, they all can be *nested*, i.e., all the intervals contain the point $\mu = 0$. The corresponding results, Theorems 2, 3 and 4, are presented in Section 6 and both formulated and proved in a context of two parameter general unfoldings.

In Section 7 we prove the invariance of certain quantities which play a very important role for the description of the dynamical phenomena at the “global resonance”.

In Section 8 we prove Lemma 2.

2 Statement of the problem and preliminary geometric constructions.

Consider a C^r -smooth ($r \geq 3$) area-preserving map f_0 satisfying the following conditions.

- A. f_0 has a saddle fixed (or periodic) point O with multipliers λ and γ , where $0 < |\lambda| < 1 < |\gamma|$ and $|\lambda\gamma| = 1$. Moreover, we will consider two different cases:
 - A.1 the saddle is *orientable*, i.e., $\lambda = \gamma^{-1}$;
 - A.2 the saddle is *non-orientable*, i.e., $\lambda = -\gamma^{-1}$.
- B. The stable and unstable invariant manifolds of the saddle O have a quadratic tangency at the points of some homoclinic orbit Γ_0 (see Figure 1(a)).

Let \mathcal{H} be a (codimension one) bifurcation manifold composed of area-preserving C^r -maps close to f_0 and such that every map of \mathcal{H} has a nontransversal homoclinic orbit close to Γ_0 . Let f_ε be a family of area-preserving C^r -maps that contains the map f_0 at $\varepsilon = 0$. We suppose that the family depends smoothly on parameters $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ and satisfies the following condition.

- C. The family f_ε is transverse to \mathcal{H} .

Let U be a small neighbourhood of the set $O \cup \Gamma_0$. Note that U consists of a small disk U_0 containing the point O and a number of small disks containing those points of Γ_0 that do not belong to U_0 (see Figure 1(a)).

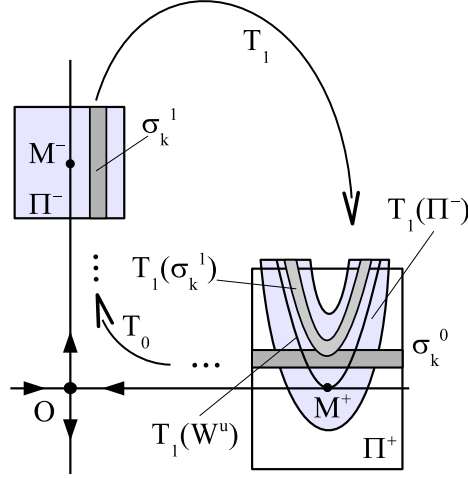


Figure 2: Geometric properties of the local and global maps T_0 and T_1 .

Definition 1. A periodic or homoclinic orbit entirely lying in U is called p -round if it has exactly p intersection points with any disk of the set $U \setminus U_0$.

In this paper we study bifurcations of *single-round* ($p = 1$) periodic orbits in the families f_ε . Note that every point of such an orbit can be considered as a fixed point of the corresponding *first return map*. Such a map is usually constructed as a superposition $T_k = T_1 T_0^k$ of two maps $T_0 \equiv T_0(\varepsilon)$ and $T_1 \equiv T_1(\varepsilon)$, see Figure 2. The map T_0 is called *local map* and it is defined as the restriction of f_ε onto U_0 , i.e., $T_0(\varepsilon) \equiv f_\varepsilon|_{U_0}$. The map T_1 is called *global map* and it is defined as $T_1 \equiv f_\varepsilon^q$ and acts from a small neighbourhood $\Pi^- \subset U_0$ of some point $M^- \in W_{loc}^u(O)$ of the orbit Γ_0 into a neighbourhood $\Pi^+ \subset U_0$ of another point $M^+ \in W_{loc}^s(O)$ of Γ_0 , where q is an integer such that $f_0^q(M^-) = M^+$. Thus, any fixed point of T_k is a point of a single-round periodic orbit for f_ε with period $k + q$. We will study maps T_k for all sufficiently large integer k . Therefore, it is very important to have good coordinate representations for both maps T_0 and T_1 .

2.1 Finite-smooth normal forms of saddle APMs.

The area-preserving map $T_0(\varepsilon)$ has a saddle fixed point O_ε for all sufficiently small ε . The simplest form for T_0 might be the linear one: $\bar{x} = \lambda(\varepsilon)x$, $\bar{y} = \gamma(\varepsilon)y$, where $|\lambda| = |\gamma|^{-1}$, however, it is non-applicable since only C^1 -linearization can be ensured here. In the real-analytical case with $\gamma = \lambda^{-1} > 0$ we can use the well-known *Birkhoff-Moser normal form* [42]

$$\begin{aligned} \bar{x} &= B(xy, \varepsilon)x = \lambda(\varepsilon)x \left(1 + \sum_{i=1}^{\infty} \beta_i(\varepsilon) \cdot (xy)^i \right), \\ \bar{y} &= B(xy, \varepsilon)^{-1}y = \lambda^{-1}(\varepsilon)y \left(1 + \sum_{i=1}^{\infty} \tilde{\beta}_i(\varepsilon) \cdot (xy)^i \right), \end{aligned} \quad (2)$$

where $B(xy, \varepsilon)$ is a real-analytic function (of the variable $u = xy$) well-defined in a small fixed neighbourhood of $u = 0$ for all sufficiently small ε . The coefficients β_i are called *Birkhoff coefficients*, the coefficients $\tilde{\beta}_i$ depend on those in such a way that $\tilde{\beta}_i$ is a single-valued functions of β_1, \dots, β_i . For example, $\tilde{\beta}_1 = -\beta_1$, $\tilde{\beta}_2 = \beta_1^2 - \beta_2$, etc.

In the smooth case, following [43, 44, 45], we can apply the so-called *finitely smooth normal forms* of the saddle map. The *main normal form* (of the first order) for the saddle map $T_0(\varepsilon)$ is given by the following lemma.

Lemma 1. [45]. *Let $T_0(\varepsilon)$ be C^r with $r \geq 3$. Then there exists a canonical C^r -change of coordinates under which $T_0(\varepsilon)$ takes the form*

$$\bar{x} = \lambda(\varepsilon)x(1 + \beta_1(\varepsilon)xy) + o(x^2y), \quad \bar{y} = \gamma(\varepsilon)y(1 - \beta_1(\varepsilon)xy) + o(xy^2), \quad (3)$$

where $\beta_1 \equiv 0$ in the case $\lambda\gamma = -1$. The change is C^{r-2} with respect to the parameters.

The following lemma concerns the n -th order normal form.

Lemma 2. *For any integer $n \geq 2$ such that $n < r/2$ (if $r = \infty$, then n is arbitrary), there exists a canonical C^{r-2n+1} change of coordinates under which $T_0(\varepsilon)$ takes the form*

$$\begin{aligned} \bar{x} &= \lambda(\varepsilon)x(1 + \beta_1(\varepsilon) \cdot xy + \dots + \beta_n(\varepsilon) \cdot (xy)^n) + o(x^{n+1}y^n), \\ \bar{y} &= \gamma(\varepsilon)y(1 + \tilde{\beta}_1(\varepsilon) \cdot xy + \dots + \tilde{\beta}_n(\varepsilon) \cdot (xy)^n) + o(x^n y^{n+1}). \end{aligned} \quad (4)$$

Moreover, in the case $\lambda\gamma = -1$, $\beta_i = \tilde{\beta}_i \equiv 0$ for odd i .

Remark 2.1. 1) We refer Lemma 1 to the paper [45], where it was proved for the case $|\lambda\gamma| = 1$ and the proof (using canonical transformations) covers also the area-preserving case, independently, whether the map $T_0(\varepsilon)$ is orientable or not. We also note that a version of Lemma 1 with the existence of a C^{r-1} -change of coordinates was proved in [1, 39] for the symplectic case and in [44] for the case $|\lambda\gamma| = 1$.

2) Note that analogous to (4) finite-smooth local normal forms for two-dimensional flows having a saddle equilibrium with eigenvalues $-\rho$ and ρ , where $\rho > 0$, were derived by E.A. Leontovich [46, 47]. When proving Lemma 2 we follow closely to the Leontovich method with some modifications proposed by V.S. Afraimovich [48].

One of the advantages of the pointed out normal forms is that they allow us to obtain a quite simple coordinate expression for the iterations T_0^k for all integer k . Namely, let $(x_i, y_i) \in U_0, i = 0, \dots, k-1$, be points such that $(x_{i+1}, y_{i+1}) = T_0(x_i, y_i)$. If T_0 is linear, then, evidently, $x_k = \lambda^k x_0, y_k = \gamma^k y_0$. We can rewrite the last formula in the so-called *cross-form* $x_k = \lambda^k x_0, y_0 = \gamma^{-k} y_k$. An analogous cross-form exists also in the nonlinear case. In the case of T_0 in the Birkhoff-Moser normal form (2), the map T_0^k can be written as follows [38]

$$x_k = \lambda^k x_0 \cdot R^{(k)}(x_0 y_k, \varepsilon), \quad y_0 = \lambda^k y_k \cdot R^{(k)}(x_0 y_k, \varepsilon), \quad (5)$$

where

$$R^{(k)}(x_0 y_k, \varepsilon) \equiv 1 + \sum_{i=1}^{\infty} \hat{\beta}_i(k) \lambda^{ik} (x_0 y_k)^i \quad (6)$$

and $\hat{\beta}_i(k)$ are some i -th degree polynomials of k with coefficients depending on β_1, \dots, β_i , in particular,

$$\hat{\beta}_1(k, \varepsilon) = \beta_1(\varepsilon) \cdot k, \quad \hat{\beta}_2(k, \varepsilon) = \beta_1^2(\varepsilon) \cdot k^2 + \beta_2(\varepsilon) \cdot k. \quad (7)$$

In the case of finitely smooth normal forms the following results hold.

Lemma 3. [45] *If T_0 takes the first order normal form (3), then T_0^k can be written as follows*

$$(x_k, y_0) = (\lambda^k x_0, \gamma^{-k} y_k)(1 + \beta_1 k \lambda^k x_0 y_k) + \lambda^{2k} (P_1(x_0, y_k, \varepsilon), Q_1(x_0, y_k, \varepsilon)) \quad (8)$$

where the functions P_1 and Q_1 are uniformly bounded along with all their derivatives up to order $(r-2)$ and the following estimates take place for the last two derivatives

$$\|(x_k, y_0)\|_{C^{r-1}} = O(|\lambda|^k), \quad \|(x_k, y_0)\|_{C^r} = o(1)_{k \rightarrow \infty}.$$

Lemma 4. [5] *If T_0 takes the n -th order normal form (4), then T_0^k can be written as*

$$\begin{aligned} x_k &= \lambda^k x_0 \cdot R_n^{(k)}(x_0 y_k, \varepsilon) + \lambda^{(n+1)k} P_n^{(k)}(x_0, y_k, \varepsilon), \\ y_0 &= \gamma^{-k} y_k \cdot R_n^{(k)}(x_0 y_k, \varepsilon) + \lambda^{(n+1)k} Q_n^{(k)}(x_0, y_k, \varepsilon), \end{aligned} \quad (9)$$

where $R_n^{(k)}(x_0 y_k, \varepsilon) = 1 + \sum_{i=1}^n \hat{\beta}_1(k) \lambda^{ik} (x_0 y_k)^i$ (a finite sum of series (6)). The functions $P_n^{(k)} = o(x_0^{n+1} y_k^n)$, $Q_n^{(k)} = o(x_0^n y_k^{n+1})$ are uniformly bounded in k along with all their derivatives with respect to x_0 and y_k up to the order $(r-2n-1)$ (and up to the order $(r-2n-1)$ with respect to the derivatives by parameters) and $\|(x_k, y_0)\|_{C^{r-2n}} = O(|\lambda|^k)$, $\|(x_k, y_0)\|_{C^{r-2n+1}} = o(1)_{k \rightarrow \infty}$.

Lemmas 1 and 3 were proved in [45] (see also [39]). Lemmas 2 and 4 were proved in [5] for the symplectic case. The proof of Lemma 4 for the non-orientable area-preserving case is practically the same and, therefore, we omit it. Thus, only Lemma 2 is really new (in its part related to the non-orientable case) and we prove it in Section 8. For the convenience of the reader we give the complete proof of this lemma considering both symplectic and non-orientable cases.

Remark 2.2. In our calculations (see e.g. the proof of Lemma 6) we will also use the *second order normal form* for T_0

$$\begin{aligned} \bar{x} &= \lambda x (1 + \beta_1 xy + \beta_2 (xy)^2) + O(|x|^3 |y|^2 (|x| + |y|)), \\ \bar{y} &= \gamma y (1 - \beta_1 xy + \tilde{\beta}_2 (xy)^2) + O(|x|^2 |y|^3 (|x| + |y|)), \end{aligned} \quad (10)$$

that is given by (4) for $n=2$, where $\tilde{\beta}_2 = \beta_1^2 - \beta_2$; in the case $\lambda\gamma = -1$ we have that $\beta_1 \equiv 0$ and $\tilde{\beta}_2 = -\beta_2$. Then formulae (9) for T_0^k with $n=2$ can be written as

$$\begin{aligned} x_k &= \lambda^k x_0 (1 + k\beta_1 \lambda^k x_0 y_k) + O(k^2 \lambda^{3k}), \\ y_0 &= \gamma^{-k} y_k (1 + k\beta_1 \lambda^k x_0 y_k) + O(k^2 \lambda^{3k}), \end{aligned} \quad (11)$$

moreover, in the case $\lambda\gamma = -1$, they take a simpler form

$$x_k = \lambda^k x_0 + O(k^2 \lambda^{3k}), \quad y_0 = \gamma^{-k} y_k + O(k^2 \lambda^{3k}). \quad (12)$$

2.2 Properties of the global map $T_1(\varepsilon)$.

In what follows, we will use in U_0 the local normal form coordinates (x, y) introduced in Section 2.1. In these coordinates both W_{loc}^s and W_{loc}^u are straightened out and, hence, we can put $M^+ = (x^+, 0)$, $M^- = (0, y^-)$, where $x^+ > 0$ and $y^- > 0$. Then the global map $T_1(\varepsilon) \equiv f^q(\varepsilon) : \Pi^- \rightarrow \Pi^+$ can be written as follows

$$\bar{x} - x^+ = F(x, y - y^-, \varepsilon), \quad \bar{y} = G(x, y - y^-, \varepsilon), \quad (13)$$

where $F(0) = 0, G(0) = 0$. Besides, one has that $G_y(0) = 0, G_{yy}(0) = 2d \neq 0$ which follows from the fact (condition B) that at $\varepsilon = 0$ the curve $T_1(W_{loc}^u) : \{\bar{x} - x^+ = F(0, y - y^-, 0), \bar{y} = G(0, y - y^-, 0)\}$ has a quadratic tangency with $W_{loc}^s : \{\bar{y} = 0\}$ at M^+ . When parameters vary this tangency can split and, moreover, we can introduce the corresponding splitting parameter $\mu \equiv G(0, 0, \varepsilon)$. By condition C, we can assume that the parameter μ belongs to the set of parameters ε . Accordingly, we can write the following Taylor expansions for the functions F and G

$$\begin{aligned} F(x, y - y^-, \varepsilon) &= ax + b(y - y^-) + e_{20}x^2 + e_{11}x(y - y^-) + e_{02}(y - y^-)^2 + \text{h.o.t.}, \\ G(x, y - y^-, \varepsilon) &= \mu + cx + d(y - y^-)^2 + f_{20}x^2 + f_{11}x(y - y^-) + f_{30}x^3 \\ &\quad + f_{21}x^2(y - y^-) + f_{12}x(y - y^-)^2 + f_{03}(y - y^-)^3 + \text{h.o.t.}, \end{aligned} \quad (14)$$

where the coefficients a, b, \dots, f_{03} (as well as x^+ and y^-) depend smoothly on ε .

In the area-preserving case, the Jacobian $J(T_1)$ of T_1 is equal identically to ± 1 for all values of ε . In particular, this implies that

$$|bc| \equiv 1 \quad \text{and} \quad R = 2ad - bf_{11} - 2ce_{02} \equiv 0, \quad (15)$$

since $J(T_1)|_{M^-} = -bc$ and $\frac{\partial J(T_1)}{\partial y}|_{M^-} = R$.

We divide the APMs under consideration into three groups:

- (i) the *symplectic maps*, when T_0 and T_1 are both orientable ($\lambda\gamma = 1$ and $bc = -1$), in this case condition A.1 holds;
- (ii) the *globally non-orientable maps*, when T_0 is orientable and T_1 is non-orientable ($\lambda\gamma = 1$ and $bc = 1$), i.e., the condition A.1 holds;
- (iii) the *locally non-orientable maps*, when T_0 is non-orientable ($\lambda\gamma = -1$), i.e., the condition A.2 holds.

Note that in the case $\lambda\gamma = -1$, the global map T_1 can be orientable ($bc = -1$) or non-orientable ($bc = 1$) depending on the choice of pairs of homoclinic points M^+ and M^- . If T_1 is orientable for a given pair (M^+, M^-) , then it is non-orientable for the pairs $(T_0(M^+), M^-)$ or $(M^+, T_0^{-1}(M^-))$ and again orientable for the pairs $(T_0^2(M^+), M^-)$, $(M^+, T_0^{-2}(M^-))$ or $(T_0(M^+), T_0^{-1}(M^-))$ etc. We will call any such a pair of the homoclinic points, when the corresponding global map T_1 is orientable, of the *needed type*. For more definiteness, we will assume that the following condition holds.

- D.** In the locally non-orientable case, we take always a pair of points $M^+ \in W_{loc}^s$ and $M^- \in W_{loc}^u$ of the homoclinic orbit Γ_0 which is of the needed type.

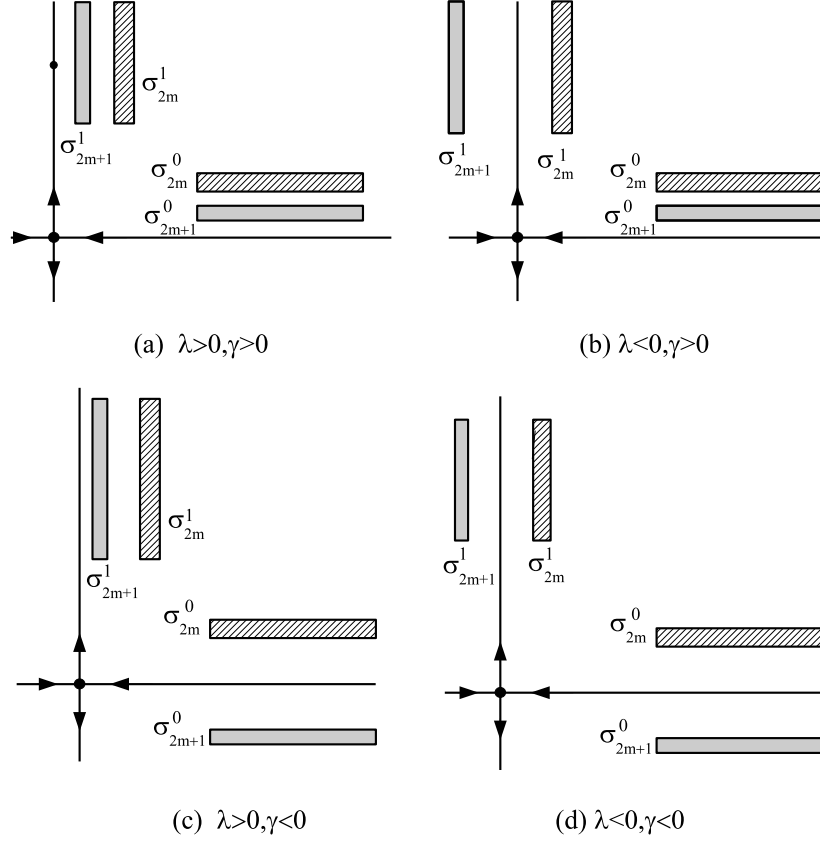


Figure 3: The strips σ_k^0 and σ_k^1 for λ and γ of various signs.

2.3 Strips, horseshoes and return maps.

We assume that the neighbourhoods Π^+ and Π^- are sufficiently small and fixed, so that $T_0(\varepsilon)(\Pi^+) \cap \Pi^+ = \emptyset$ and $T_0^{-1}(\varepsilon)(\Pi^-) \cap \Pi^- = \emptyset$ for all small ε . Then the domain of definition of the map from Π^+ to Π^- under iterations of $T_0(\varepsilon)$ consists of infinitely many nonintersecting strips σ_k^0 belonging to Π^+ and accumulating to $W_{loc}^s \cap \Pi^+$ as $k \rightarrow \infty$. Analogously, the range of this map consists of infinitely many (nonintersecting) strips $\sigma_k^1 = T_0^k(\sigma_k^0)$ belonging to Π^- and accumulating to $W_{loc}^u \cap \Pi^-$ as $k \rightarrow \infty$. See Figure 3 where a location of the strips is shown for various cases of the signs of λ and γ .

According to (13) and (14), the images $T_1(\sigma_j^1)$ of the strips σ_j^1 have a horse-shoe form and accumulate to the curve $l_u = T_1(W_{loc}^u)$ as $j \rightarrow \infty$. Note that any orbit staying entirely in U must intersect both the neighbourhoods Π^- and Π^+ (otherwise, it would not be close enough to $\bar{\Gamma}_0$). Thus, such orbits must have points belonging to the intersections of the horseshoes $T_1(\sigma_j^1)$ and the strips σ_i^0 for all possible integer i and j .

When μ varies, the location of the horseshoes $T_1(\sigma_j^1)$ changes: they move together with $T_1(W_{loc}^u)$. It implies that the character of mutual intersections of the strips and horseshoes can change drastically. This concerns, in particular, the strips σ_i^0 and horseshoes $T_1(\sigma_i^1)$ with the same numbers i . Thus, when μ changes, bifurcations of Smale horseshoes creation/destruction will occur. In order to understand these bifurcations we need to study, first of all, the dynamics of the map f_0 , i.e., at $\mu = 0$.

For this goal, we study in the next section the semi-local dynamics of the APMs with the

homoclinic tangencies under conditions **A** and **B**.

3 On a semi-local dynamics of APMs with homoclinic tangencies.

In this section we consider APMs f_0 satisfying the conditions **A** and **B**. The main goal is to understand the semi-local dynamics of f_0 , i.e., the structure of the set N of orbits of the map f_0 entirely lying in a small fixed neighbourhood U of the contour $O \cup \Gamma_0$. Since U is actually small and contains the neighbourhoods Π^+ and Π^- of the homoclinic points M^+ and M^- , we can assume that, apart from the orbit O , the set N contains only such orbits that have intersection points with both Π^+ and Π^- . Equivalently, for a given sufficiently large integer $\bar{k} > 0$, we can assume that the neighbourhoods Π^+ and Π^- contain the strips σ_k^0 and σ_k^1 , respectively, only for $k \geq \bar{k}$. In other words, we will consider only such orbits entirely lying in U whose points from Π^+ can reach Π^- after a number of iterations (under f_0) that is not less than \bar{k} . We denote the set of such orbits by $N_{\bar{k}} \equiv N_{\bar{k}}(f_0)$.

We study properties of the orbits in $N_{\bar{k}}(f_0)$ using the main analytical result, Lemma 5, (proved in [43, 49]) as a tool for detecting the type of intersection between the horseshoes $T_1(\sigma_j^1)$ and strips σ_i^1 for various $i, j \geq \bar{k}$. We assume that the initial tangency (under the conditions **A** and **B**) does not split: this corresponds to $\mu = 0$ in (14). We will show that, in this case, the set $N_{\bar{k}}$ can be completely described (in terms of symbolic dynamics) for an open and dense set⁴ of maps from \mathcal{H} . The density should be regarded in the following sense: for a given \bar{k} , in \mathcal{H} there exists an open set of maps whose set $N_{\bar{k}}$ is completely determined and this set becomes dense as $\bar{k} \rightarrow \infty$: in fact, one has to exclude only maps satisfying certain conditions like “the invariant τ , given in (23), is integer number”.

3.1 Conditions for the intersection of horseshoes and strips.

Evidently, any orbit of $N_{\bar{k}}$ (except for the orbits O and Γ_0) must have points belonging to the intersection of the horseshoes $T_1(\sigma_j^1)$ and strips σ_i^0 for some $i, j \geq \bar{k}$. Thus, the structure of $N_{\bar{k}}$ depends essentially on the character of this intersection.

Definition 2. *We say that the horseshoe $T_1(\sigma_j^1)$ has a regular intersection with the strip σ_i^0 if*

- (i) *the set $T_1(\sigma_j^1) \cap \sigma_i^0$ consists of two connected components Δ_{ij}^1 and Δ_{ij}^2 ;*
- (ii) *the map $T_1 T_0^j$ restricted to the preimage $(T_1 T_0^j)^{-1} \Delta_{ij}^\alpha \subset \sigma_j^0$ of the component Δ_{ij}^α , where $\alpha = 1, 2$, is a saddle map (i.e., it is exponentially contracting along one of coordinates, x , and expanding along the other one, y), see Figure 4.*

The following lemma provides sufficient conditions characterizing intersections of the strips and horseshoes.

Lemma 5. [43, 49] *Given f_0 satisfying conditions **A** and **B**, with c and d given in (14) at $\mu = 0$, there exist a constant $S_1 > 0$ and a sufficiently large integer \bar{k} such that, for any $i, j \geq \bar{k}$, the following assertions hold:*

⁴This is not the case if $|\lambda\gamma| \neq 1$: as it is shown in [50, 51]), systems with infinitely degenerate periodic orbits are dense among those with quadratic homoclinic tangencies.

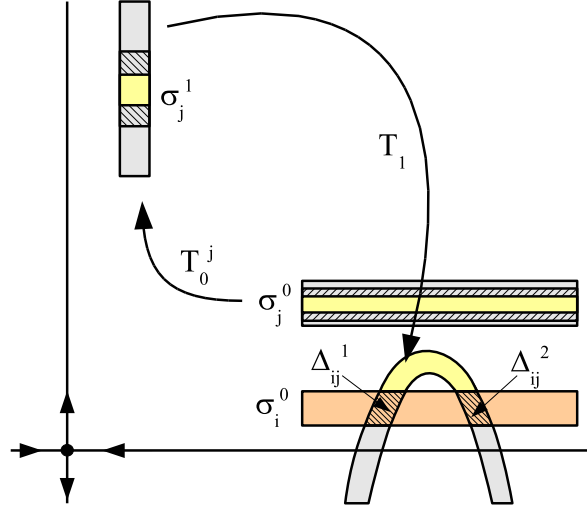


Figure 4: Regular intersection of the horseshoe $T_1(\sigma_j^1)$ and the strip σ_i^0 (Definition 2).

(i) If

$$d(\gamma^{-i}y^- - c\lambda^jx^+) > S_1(|\lambda|^i + |\lambda|^j)|\lambda|^{\bar{k}/2}, \quad (16)$$

then the horseshoe $T_1(\sigma_j^1)$ and strip σ_i^0 intersect regularly.

(ii) If

$$d(\gamma^{-i}y^- - c\lambda^jx^+) < -S_1(|\lambda|^i + |\lambda|^j)|\lambda|^{\bar{k}/2}, \quad (17)$$

then $T_1(\sigma_j^1) \cap \sigma_i^0 = \emptyset$.

It is convenient to reformulate this lemma as follows:

(i') if the horseshoe $T_1(\sigma_j^1)$ has an irregular intersection with the strip σ_i^0 (i.e., the intersection $T_1(\sigma_j^1) \cap \sigma_i^0$ consists of one connected component or the corresponding maps from Definition 2 are not saddle), then the following inequality holds

$$|d||\gamma^{-i}y^- - c\lambda^jx^+| \leq S_1(|\lambda|^i + |\lambda|^j)|\lambda|^{\bar{k}/2}, \quad (18)$$

(ii') if $T_1(\sigma_j^1) \cap \sigma_i^0 \neq \emptyset$, then the following inequality holds

$$d(\gamma^{-i}y^- - c\lambda^jx^+) \geq -S_1(|\lambda|^i + |\lambda|^j)|\lambda|^{\bar{k}/2}. \quad (19)$$

The inequalities (16)–(19) have a rather simple geometrical sense. The strip σ_i^0 is a narrow horizontal rectangle in Π^+ having a central line $y = \gamma^{-i}y^-$, while, the strip σ_j^1 is a narrow vertical rectangle in Π^- having a central line $x = \lambda^jx^+$. By (13) and (14), the strip σ_j^1 is mapped under T_1 into a horseshoe which contains a parabola $y = c\lambda^jx^+ + d(x - x^+)^2/b^2$. The inequality $d(\gamma^{-i}y^- - c\lambda^jx^+) > 0$ means that the straight line $y = \gamma^{-i}y^-$ and the parabola are crossed in two points, whereas, the inequality $d(\gamma^{-i}y^- - c\lambda^jx^+) < 0$ implies that these curves do not intersect. The small coefficient in the right side of the inequalities appears in order to take into account a non-zero thickness of the strips and horseshoes.

Note that when the regular intersection exists one can establish certain hyperbolic properties of (area-preserving) maps f_0 . The following simplest result of such kind relates to the existence of Smale horseshoes in the first return maps T_i .

Proposition 3.1. [43] *Given f_0 satisfying conditions **A** and **B**, assume that the strip σ_i^0 and the horseshoe $T_1(\sigma_i^1)$ (with the same number) have a regular intersection for which condition (16) with $i = j$ holds. Then the first return map $T_i \equiv T_1 T_0^i : \sigma_i^0 \mapsto \sigma_i^0$ is a Smale horseshoe map, i.e., the map T_i has a non-wandering set Ω_i which is the closed invariant uniformly hyperbolic set such that the system $T_i|_{\Omega_i}$ is conjugate to the topological Bernoulli shift with two symbols.*

3.2 Three classes of APMs with homoclinic tangencies.

Clearly, the structure of integer solutions of the inequalities (16)–(19) depends, first of all, on the signs of the parameters λ, γ, c and d . This means that the structure of the set N_k depends essentially on the type of the homoclinic tangency. By this principle, the same as for the case of general diffeomorphisms [11, 43, 52], we can divide quadratic homoclinic tangencies in the area-preserving case into three classes in the following way:

- *The first class* is connected to the tangencies with $\lambda > 0, \gamma = \lambda^{-1}, c < 0$ and $d < 0$.
- *The second class* has to do with the tangencies with $\lambda > 0, \gamma = \lambda^{-1}, c < 0$ and $d > 0$.
- The tangencies of all other types (with all other combinations of the signs of λ, γ, c and d) belong to *the third class*.

We will say also that a given APM is of the first, second or third class, if it has a homoclinic tangency of the first, second or third class, respectively.

Concerning maps of the third class, one can obtain formally 14 different combinations of the signs of coefficients λ, c and d . However, some of them can be transformed to the others.

For example, if we choose the pair $M^{+'} = T_0(M^+)$ and M^- of homoclinic points instead of M^+ and M^- , the new global map $T_1' = T_1 T_0 : \Pi^- \rightarrow T_0(\Pi^+)$. Then, by (3), (13) and (14), it can be written as follows

$$\bar{x} = \lambda x^+ + \lambda a x + \lambda b(y - y^-) + \dots, \quad \bar{y} = \gamma c x + \gamma d(y - y^-)^2 + \dots.$$

If λ is positive, we have that $x^{+'} = \lambda x^+, c' = \gamma c$ and $d' = \gamma d$. If λ is negative, first we make the change $x \mapsto -x$ and then obtain that $x^{+'} = -\lambda x^+ > 0, c' = -\gamma c$ and $d' = \gamma d$. Thus, in both cases we can write that

$$\text{sign } c' = \text{sign } (c\lambda\gamma), \quad \text{sign } d' = \text{sign } (d\gamma) \quad (20)$$

and, hence, in the case $\lambda = \gamma^{-1} < 0$, by (20), we can always assume $d > 0$.

Besides, in the area-preserving case, there is no necessity to distinguish f_0 and f_0^{-1} . Moreover, the following relations take place for the local and global maps $\tilde{T}_0 = T_0^{-1}$ and $\tilde{T}_1 = T_1^{-1}$ of f_0^{-1}

$$\tilde{\lambda} = \gamma^{-1}, \quad \tilde{\gamma} = \lambda^{-1}, \quad \tilde{c} = \frac{1}{c}, \quad \tilde{d} = -\frac{d}{cb^2}. \quad (21)$$

and, indeed, by (13) and (14), the map $\tilde{T}_1 = T_1^{-1}$ can be written as

$$y - y^- = \frac{1}{b}(\bar{x} - x^+) + \dots, \quad x = \frac{1}{c}\bar{y} - \frac{d}{b^2 c}(\bar{x} - x^+)^2 + \dots,$$

which takes the standard form (13) if we interchange the variables x and y as well as the constants x^+ and y^- . Thus, we will not distinguish the combinations $\lambda > 0, \gamma > 0, c > 0, d > 0$ and $\lambda > 0, \gamma > 0, c > 0, d < 0$ (see Figure 5). Also, in the case $\lambda\gamma = -1$, we can set $\lambda < 0, \gamma > 0$.

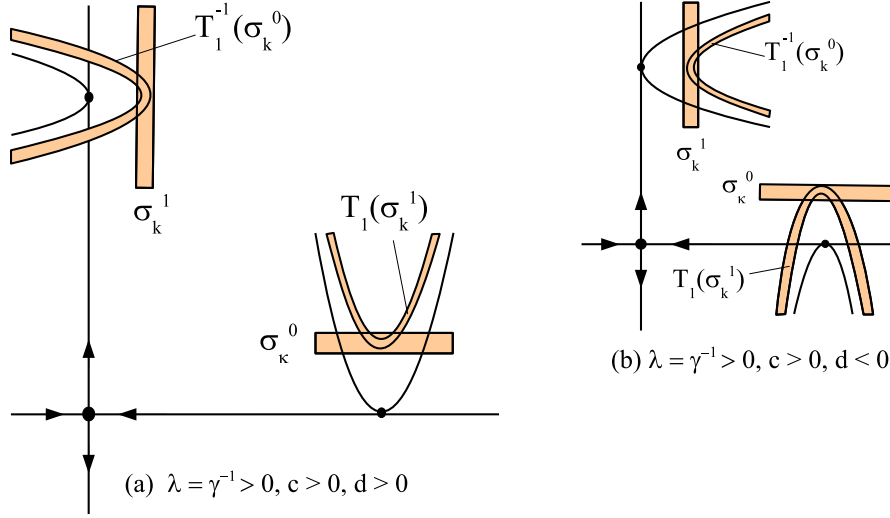


Figure 5: APMs with a homoclinic tangency of the third class for $\lambda = \gamma^{-1} > 0$: (a) the case $c > 0, d > 0$; (b) the case $c > 0, d < 0$. One can see the complete analogy in the geometric structure of the strips and horseshoes in these cases, especially if we consider the map f_0^{-1} in case (b).

Therefore, we can reduce the number of different types of homoclinic tangencies of the third class to the 5 main different ones represented in Figure 6. We denote by H_3^i , $i = 1, \dots, 5$, the corresponding locally connected codimension 1 bifurcation surfaces of APMs with homoclinic tangencies. Note that in the locally non-orientable case, $\lambda\gamma = -1$, we have to consider always, by condition **D** of Section 2.2, pairs (M^+, M^-) of the homoclinic points of *the needed type* (the corresponding global map T_1 is orientable, i.e., $bc = -1$). This means that the sign of c plays an important rôle and, thus, the surfaces H_3^2 and H_3^3 split into the parts $H_3^{2,1}$, $H_3^{2,2}$ and $H_3^{3,1}$, $H_3^{3,2}$, respectively, see the table of Figure 6. However, we note that the semi-local dynamics of the maps in $H_3^{2,1}$ and $H_3^{2,2}$ or in $H_3^{3,1}$ and $H_3^{3,2}$ are “of the same type” and only the corresponding invariant sets (e.g. Smale horseshoes Ω_i), when they exist, will be orientable and non-orientable, respectively, see Figure 9 below.

	H_3^1	H_3^2		H_3^3		H_3^4	H_3^5
		$H_3^{2,1}$	$H_3^{2,2}$	$H_3^{3,1}$	$H_3^{3,2}$		
λ	+	-	-	-	-	-	-
γ	+	+	+	+	+	-	-
d	$+(-)$	-	-	+	+	$+(-)$	$+(-)$
c	+	+	-	+	-	+	-

Figure 6: Five types of APMs of the third class.

Note that symplectic maps can arise in H_3^1, H_3^4 and H_3^5 , while maps in H_3^2 and H_3^3 are always orientation-reversing. Besides, non-orientable APMs can have homoclinic tangencies of all types (in this case, the Jacobian of T_1 is negative for homoclinic tangencies of the first and second classes as well as for maps inside H_3^1, H_3^4 and H_3^5).

3.3 Dynamical properties of APMs of the first and second classes.

For maps of the first class, the inequality (17) holds for all $i, j \geq \bar{k}$. It follows that $T_1(\sigma_j^1) \cap \sigma_i^0 = \emptyset$ for all sufficiently large i and j (see Figure 7(a)) which implies the following result:

Proposition 3.2. [43] *Let f_0 be an APM of the first class. Then there exists such sufficiently large \bar{k} that the set $N_{\bar{k}}$ has the trivial structure: $N_{\bar{k}} = \{O, \Gamma_0\}$.*

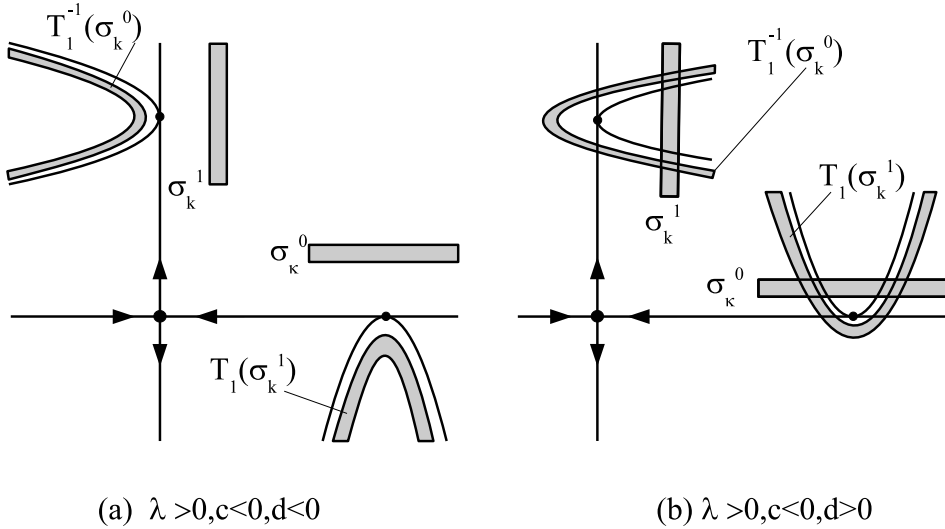


Figure 7: APMs with a homoclinic tangency a) of the first class; b) of the second class.

For maps of the second class, inequality (16) holds for all $i, j \geq \bar{k}$. This means that all the horseshoes $T_1(\sigma_j^1)$ and strips σ_i^0 have regular intersection. Therefore, the set $N_{\bar{k}}$ possesses a non-uniformly hyperbolic structure and all orbits in $N_{\bar{k}}$, except Γ_0 , are saddle (see also [43]). Moreover, we can give the exact description of the set $N_{\bar{k}}$. Namely, let $B_{\bar{k}+q}^3$ be a subsystem of the topological Bernoulli scheme (shift) with three symbols $(0, 1, 2)$ consisting only of (bi-infinite) sequences of the form

$$(\dots, 0, \alpha_{s-1}, \overbrace{0, \dots, 0}^{k_s+q}, \alpha_s, \overbrace{0, \dots, 0}^{k_{s+1}+q}, \alpha_{s+1}, 0, \dots), \quad (22)$$

where $\alpha_s \in \{1, 2\}$, $k_s \geq \bar{k}$ for any s and any sequence (22) does not contain two neighboring nonzero symbols. We assume also that the set of sequences (22) includes sequences having strings of infinite length composed by zeros. Let $\tilde{B}_{\bar{k}+q}^3$ be a factor-system resulted from $B_{\bar{k}+q}^3$ by identifying homoclinic orbits $(\dots, 0, \dots, 0, 1, 0, \dots, 0, \dots)$ and $(\dots, 0, \dots, 0, 2, 0, \dots, 0, \dots)$. We denote this orbit by $\tilde{\omega}$ as well as the orbit $(\dots, 0, \dots, 0, \dots)$ by \tilde{O} .

Proposition 3.3. [43, 2] *Let f_0 be a map of the second class. Then, for any sufficiently large \bar{k} , the system $f_0|_{N_{\bar{k}}}$ is topologically conjugate to $\tilde{B}_{\bar{k}+q}^3$. Moreover, the conjugating homeomorphism h is such that $h(\Gamma_0) = \tilde{\omega}$ and $h(O) = \hat{O}$.*

3.4 Dynamical properties of APMs of the third class.

Consider the following number

$$\tau = \frac{1}{\ln |\lambda|} \ln \left| \frac{cx^+}{y^-} \right|. \quad (23)$$

It was shown in [43] that τ is invariant on two-dimensional diffeomorphisms with a homoclinic tangency to a neutral saddle (i.e., condition $|\lambda\gamma| = 1$ holds but the diffeomorphism itself is not necessarily area-preserving). We will show (see also [43, 2, 3]) that τ can be effectively used for the description of dynamics of orbits from a small neighbourhood of the homoclinic orbit in the case of APMs of the third class.

We will use below the notations $[\tau]$ and $\{\tau\}$ for the integer and fractional part of τ , respectively.

3.4.1 APMs in H_3^1 .

First, we consider maps in H_3^1 , i.e., APMs with $\lambda = \gamma^{-1} > 0$ and $c > 0$ corresponding to the first column of the table of Figure 6. For more definiteness, we assume that d is positive (the case $d < 0$ is reduced to this for f_0^{-1} , see Section 3.2).

Proposition 3.4. *Let $f_0 \in H_3^1$ and $\tau \neq 0$. Then there exists an integer number $\bar{k} = \bar{k}(\tau)$ such that $\bar{k}(\tau) \rightarrow \infty$ as $\tau \rightarrow 0$ and the following holds.*

- 1) *If $\tau < 0$, then the set $N_{\bar{k}}$ has a trivial structure: $N_{\bar{k}} = \{O, \Gamma_0\}$.*
- 2) *If $\tau > 0$, the set $N_{\bar{k}}$ contains a nontrivial hyperbolic subset including infinitely many horseshoes Ω_i for all $i \geq \bar{k}$.*

Proof. 1) Consider the inequality (17) which can be rewritten as

$$\lambda^i \left(y^- + \frac{S_1}{d} \lambda^{\bar{k}/2} \right) < \lambda^j \left(cx^+ - \frac{S_1}{d} \lambda^{\bar{k}/2} \right),$$

since $\lambda = \gamma^{-1} > 0$, $c > 0$ and $d > 0$. Taking logarithms of both sides, we obtain the inequality

$$j - i + \tau < -S|\lambda|^{\bar{k}/2}, \quad (24)$$

where S is a positive constant (independent of i, j and \bar{k}). By Lemma 5, if $i \geq \bar{k}$ and $j \geq \bar{k}$ satisfy (24), then $T_1(\sigma_j^1) \cap \sigma_i^0 = \emptyset$. Note that the inequality (24) with $\tau < 0$ has all the solutions of the form $i \geq j$. Since $d > 0$, this means that, for all $k \geq \bar{k}$, the horseshoes $T_1\sigma_k^1$ are located above the *own* strips σ_k^0 , see Figure 8(a). Therefore, all orbits, except for O and Γ_0 , leave the neighbourhood U under forward iterations of f_0 .

2) Consider the inequality (16) which can be written now in the form

$$j - i + \tau > S|\lambda|^{\bar{k}/2}. \quad (25)$$

When τ is positive this inequality, for sufficiently large \bar{k} , has always infinitely many integer solutions of the form $j \leq i$ including the solutions $j = i$. By Lemma 5, this means that the horseshoes $T_1\sigma_i^1$ have regular intersection with the strips σ_i^0 , see Figure 8(b). By Proposition 3.1, this implies that if $\tau > 0$, the map $f_0 \in H_3^1$ has infinitely many Smale horseshoes Ω_i . \square

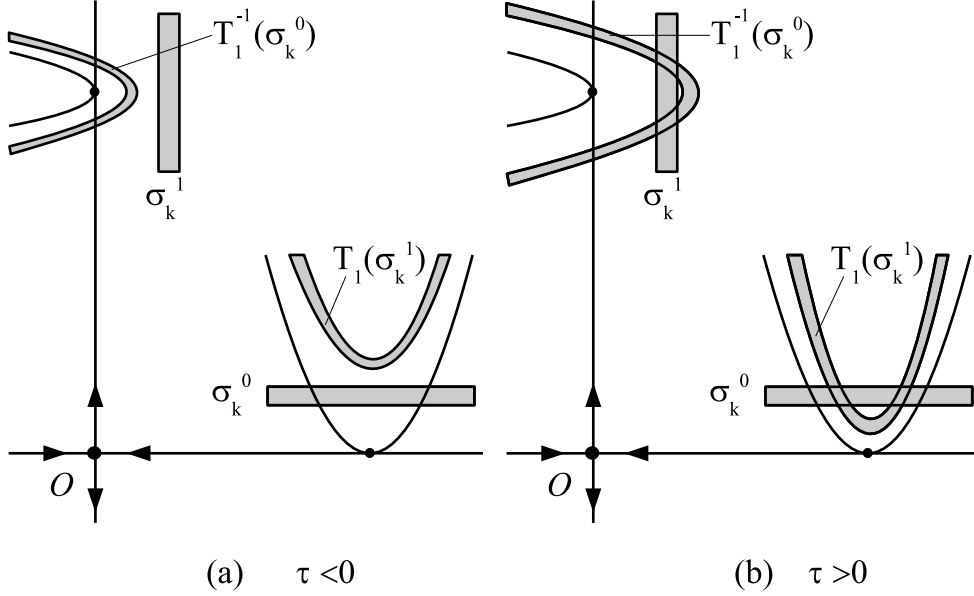


Figure 8: A creation of the Smale horseshoes Ω_k at transition from $\tau < 0$ to $\tau > 0$ in the case of APMs in H_3^1 .

Proposition 3.5. *Let $f_0 \in H_3^1$ and $\tau > 0$. If τ is not integer, then for some $\bar{k} = \bar{k}(\tau)$, where $\bar{k}(\tau) \rightarrow \infty$ as $\{\tau\} \rightarrow 0$, the set $N_{\bar{k}}$ is completely described in terms of symbolic dynamics.*

Proof. Taking logarithm of both sides, the inequality (18) is rewritten as follows

$$|j - i + \tau| \leq S|\lambda|^{\bar{k}/2}. \quad (26)$$

If τ is not an integer, this inequality has no integer solutions for sufficiently large $\bar{k} = \bar{k}(\tau)$. Thus, in this case all the strips and horseshoes have either regular or empty intersections. Taking into account only the regular intersections we obtain the complete description for $N_{\bar{k}}$. \square

The description of the set $N_{\bar{k}}$ from Proposition 3.5 can be obtained as follows. If $\tau > 0$ and $\{\tau\} \neq 0$, we can write the inequality (25) in the equivalent form

$$j - i + [\tau] + \frac{1}{2} > 0, \quad i, j \geq \bar{k}, \quad (27)$$

where $\bar{k} = \bar{k}(\tau)$ is sufficiently large (in any case, $\bar{k}(\tau) \rightarrow \infty$ as $\{\tau\} \rightarrow 0$). Let $\mathcal{B}_{[\tau]}^3$ be a subsystem of $\tilde{B}_{\bar{k}+q}^3$ containing the orbits \hat{O} , $\tilde{\omega}$ and all orbits corresponding to the sequences (22) in which

- all the integer numbers k_s and k_{s+1} satisfy the inequality (27) with $k_s = j$, $k_{s+1} = i$ including also all the pairs k_0, k_1 where $k_0 = \infty$, $\bar{k} \leq k_1 < \infty$.

Then, using methods of [43, 49], we prove the following result

Proposition 3.6. *Assume that the hypotheses of Proposition 3.5 hold. Then the system $f_0|_{N_{\bar{k}}}$ is topologically conjugate to $\mathcal{B}_{[\tau]}^3$.*

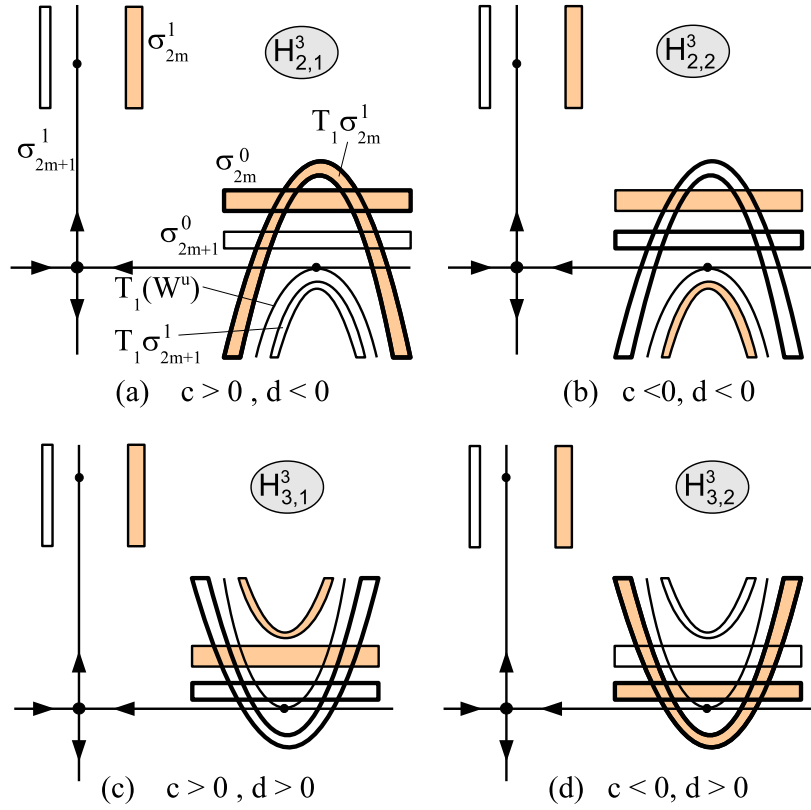


Figure 9: Geometry of the strips and horseshoes for the locally non-orientable case: for APMs maps in H_3^2 (above) and H_3^3 (below). It is illustrated that the Smale horseshoes Ω_k for the maps in $H_{3,1}^2$ (a) and $H_{3,1}^2$ (b) as well as in $H_{3,1}^3$ (c) and $H_{3,2}^3$ (d) have different orientation.

3.4.2 APMs of the third class with negative λ .

For maps of the third class with negative λ , a quick glance at the Figures 9 and 10 suggests us that the description of $N_{\bar{k}}$ has several peculiarities in each of the six cases under consideration. Moreover, since λ is negative, it is clear that this description must include some conditions on the parity of the numbers i and j of the strips σ_i^0 and the horseshoes $T_1(\sigma_j^1)$.

Note that for the maps inside H_3^3 and H_3^5 the set $N_{\bar{k}}$ has always a nontrivial structure. In particular, the following result (which is quite analogous to the Proposition 3.3) holds.

Proposition 3.7. *Let $f_0 \in H_3^3, H_3^5$. Then there exists \bar{k} such that the set $N_{\bar{k}}$ contains a non-uniformly hyperbolic subset $\hat{N}_{\bar{k}}$ and the following holds*

1. if $f_0 \in H_{3,1}^3$, then $f_0|_{\hat{N}_{\bar{k}}}$ is conjugate to B_{k+q}^{3odd} ,
2. if $f_0 \in H_{3,2}^3 \cup H_{3,1}^5$, then $f_0|_{\hat{N}_{\bar{k}}}$ is conjugate to B_{k+q}^{3ev} ,

where B_{k+q}^{3odd} (respectively, B_{k+q}^{3ev}) is a subsystem of the system B_{k+q}^3 consisting only of bi-infinite sequences of the form (22) where all numbers k_s are odd (respectively, even).

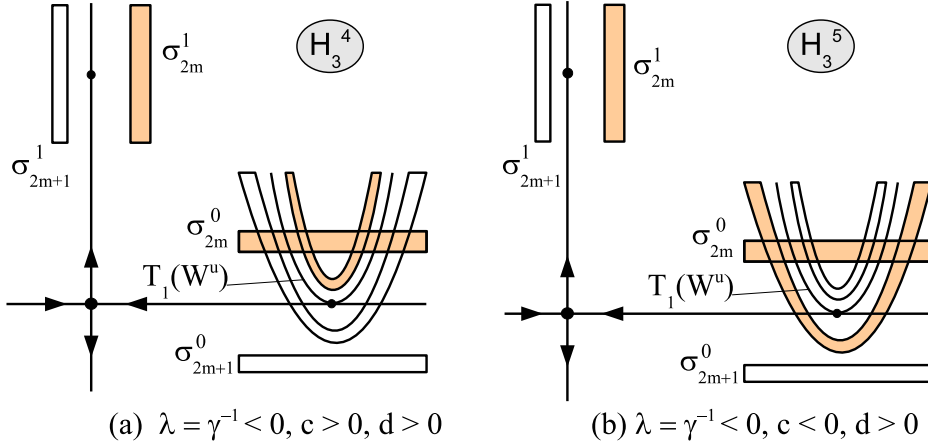


Figure 10: Geometry of the strips and horseshoes for APMs in (a) H_3^4 and (b) H_3^5 .

In particular, it follows from Proposition 3.7 that any map $f_0 \in H_3^3, H_3^5$ has always infinitely many Smale horseshoes Ω_i with numbers i running all sufficiently large integers of a certain parity:⁵ all the numbers are odd for $f_0 \in H_{3,1}^3$ and even for $f_0 \in H_{3,2}^3 \cup H_3^5$, see Figure 9(c)–(d) and Figure 10(b).

However, as for the maps in H_3^1 , the horseshoe geometry of APMs in $H_3^i, i = 2, 3, 4$ depends essentially on τ and, above all, on the sign of τ that one can see in Figure 11. Below we consider some results analogous to those for maps in H_3^1 .

Proposition 3.8. *Let $f_0 \in H_3^2$. 1) If $\tau > 0$, then there exists $\bar{k} = \bar{k}(\tau)$, $\bar{k}(\tau) \rightarrow \infty$ as $\tau \rightarrow +0$, such that the set $N_{\bar{k}}$ has a trivial structure: $N_{\bar{k}(\tau)} = \{O, \Gamma_0\}$. 2) If $\tau < 0$, the set $N_{\bar{k}}$ contains infinitely many horseshoes Ω_i , where the numbers i are even for $f_0 \in H_3^{2,1}$ (see Figure 11) or odd for $f_0 \in H_3^{2,2}$.*

Proof. Consider the case of $f_0 \in H_3^{2,1}$ (the proof for the case of $f_0 \in H_3^{2,2}$ is analogous).

1) As one can see in Figure 9(a), all horseshoes $T_1(\sigma_j^1)$ with odd j do not intersect any strip $\sigma_i^0 \subset \Pi^+$: here the inequality (17) holds for any i when j is odd (since $\lambda < 0, \gamma > 0, c > 0, d < 0$). This means that only the strips σ_i^0 and the horseshoes $T_1(\sigma_j^1)$ with both even i and j can be responsible for a nontrivial structure of $N_{\bar{k}}(f_0)$. Now we assume that i and j are even and consider the inequality (19) that can be rewritten as

$$|\lambda|^i \left(y^- - |d|^{-1} S_1 |\lambda|^{\bar{k}/2} \right) \leq |\lambda|^j \left(|c| x^+ + |d|^{-1} S_1 |\lambda|^{\bar{k}/2} \right). \quad (28)$$

Taking logarithm of both sides of (28), we obtain the inequality

$$j - i + \tau \leq -S |\lambda|^{\bar{k}/2}, \quad i, j = 0 \pmod{2}. \quad (29)$$

If τ is positive, (29) has only (integer) solutions (i, j) such that $i > j$. This means that any horseshoe $T_1(\sigma_j^1)$ can only intersect the strips σ_i^0 whose number i is greater than j . Since $d < 0$, this means that the backward semi-orbit of any point from Π^+ (except for M^+) leaves U .

⁵ The horseshoes are orientable for $f_0 \in H_3^{3,2}$ and non-orientable for $f_0 \in H_3^{3,1}$, when $f_0 \in H_3^5$ the horseshoes are orientable in the symplectic case and non-orientable in the globally non-orientable case.

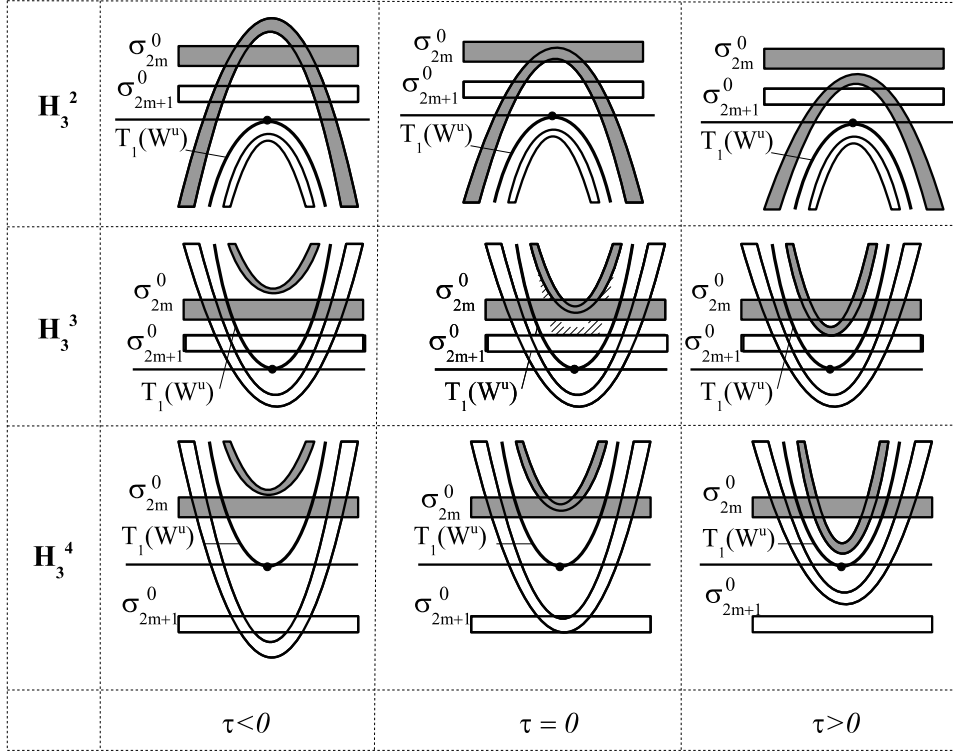


Figure 11: A creation/destruction of the new Smale horseshoes Ω_{2m} and/or Ω_{2m+1} at transition from $\tau < 0$ to $\tau > 0$ in the case of APMs in $H_3^{2,1}$, $H_3^{3,1}$ and H_3^4 .

2) Let τ be negative. Consider now the inequality (16) for even i and j which can be rewritten as follows

$$j - i + \tau < S|\lambda|^{\bar{k}/2}, \quad i, j = 0(\text{mod}2). \quad (30)$$

Since $\tau < 0$, this inequality has infinitely many integer solutions of the form $i \leq j$, in particular, it has the solutions $i = j$ with even i . This implies, by Proposition 3.1, the existence of infinitely many horseshoes Ω_i , where i runs all sufficiently large even integer numbers. \square

In the same way as was done in Proposition 3.5, one can give a complete description of the set $N_{\bar{k}(\tau)}$ in the case $f_0 \in H_3^2$ when τ is negative and not integer. Namely, let $\mathcal{B}_{[\tau],2}^3$ be a subsystem of $\tilde{\mathcal{B}}_{k+q}^3$ containing the orbits \hat{O} and $\tilde{\omega}$ and such that, in any sequence (22),

- every k_s is even
- the numbers k_s and k_{s+1} satisfy the inequality $k_s - k_{s+1} + [\tau] + \frac{1}{2} < 0$, including such codings with $k_{-1} < \infty, k_0 = \infty$.

Proposition 3.9. *Let $f_0 \in H_3^2$ and $\tau < 0$ be not integer. Then, for some $\bar{k} = \bar{k}(\tau) \rightarrow \infty$ as $\{\tau\} \rightarrow 0$, the system $f_0|_{N_{\bar{k}}}$ is topologically conjugate to $\mathcal{B}_{[\tau],2}^3$.*

Note that Proposition 3.7 deals with horseshoes Ω_i which exist always for maps in H_3^3 and H_3^5 . However, other horseshoes can appear here when varying τ that the following result shows for maps in H_3^3 .

Proposition 3.10. *If $\tau < 0$, then any map $f_0 \in H_{3,1}^3$ (respectively, $f_0 \in H_{3,2}^3$) has no horseshoes Ω_i with sufficiently large even i (respectively, with odd i), whereas for any $\tau > 0$ infinitely many such horseshoes exists.*

The proof is quite analogous to the proof of Proposition 3.8 and we omit it.

Concerning maps in H_3^4 , we note that they have a specific peculiarity related to the fact that the value $\tau = 0$ is here a “distinctive switch” between the strips σ_0^i and horseshoes $T_1(\sigma_j^1)$ involved in the dynamics, since they only occur for even i and j when $\tau > 0$ and for odd i and j when $\tau < 0$. Moreover, in this case the dynamics can be trivial only in the case $\tau = 0$. The corresponding result can be formulated as follows.

Proposition 3.11. *Let $f_0 \in H_3^4$ and $\tau \neq 0$. Then there exists an integer $\bar{k} = \bar{k}(\tau) \rightarrow \infty$ as $\tau \rightarrow 0$ such that the following holds. If $\tau > 0$ (respectively, $\tau < 0$), the set $N_{\bar{k}}$ contains infinitely many horseshoes Ω_i , where $i \geq \bar{k}$ runs for all sufficiently large even integers (respectively, odd integers). Moreover, $N_{\bar{k}}$ does not contain any orbit having intersection points with some strip σ_j^0 for odd j (respectively, for even j).*

Proof. Consider inequality (16) for even i and j . Since $c > 0, d > 0, \lambda = \gamma^{-1} < 0$, it can be rewritten as $|d|(|\lambda|^i y^- - |c||\lambda|^j x^+) > S_{ij}(\bar{k})$ or, accordingly, as

$$j - i + \tau > S|\lambda|^{\bar{k}/2}, \quad i, j = 0(\text{mod } 2). \quad (31)$$

Clearly, if $\tau > 0$ and \bar{k} is sufficiently large, this inequality has infinitely many integer solutions with $j \leq i$ and, in particular, with $i = j$. This implies, by Proposition 3.1, that infinitely many horseshoes Ω_i with even i exist in $N_{\bar{k}}$.

The inequality (16) for odd i and j can be written as $|d|(-|\lambda|^i y^- + |c||\lambda|^j x^+) > S_{ij}(\bar{k})$ or as

$$j - i + \tau < -S|\lambda|^{\bar{k}/2}, \quad i, j = 1(\text{mod } 2).$$

If $\tau < 0$, this inequality has infinitely many integer solutions of the form $j \geq i$ including $j = i$. This implies that (when $\tau < 0$) any map $f_0 \in H_3^4$ has infinitely many horseshoes Ω_i with odd i .

For $f_0 \in H_3^4$ we have always that $T_1(\sigma_j^1) \cap \sigma_i^0 = \emptyset$ for even j and odd i , since the inequality (17) holds here (see also Figure 10(a)). Let $\tau > 0$. We consider some strip σ_j^0 with odd j . Then the horseshoe $T_1(\sigma_j^1)$ can intersect only those strips σ_i^0 with odd numbers i satisfying the inequality (19) which is equivalent to the inequality $j - i + \tau \leq S|\lambda|^{\bar{k}/2}$ for odd i and j . Evidently, the last inequality has integer solutions only of the form $i > j$, since $\tau > 0$. Thus, there are no points on σ_j^0 which can return back to σ_j^0 after forward iterations by f_0 . Moreover, under backward iterations all points of σ_j^0 leave U , since if $T_1^{-1}(\sigma_j^0) \cap \sigma_l^0 \neq \emptyset$, then l is odd and $l < i$. Thus, if $\tau > 0$, only the strips σ_k^0 with even k can contain points of orbits from $N_{\bar{k}}$. The case $\tau < 0$ is proved similarly. \square

Concerning the maps in H_3^5 , we note that they are not sensitive to the resonance $\tau = 0$ and, moreover, like APMs of the second class, the set $N_{\bar{k}}$ admits here a complete description when $\tau \in (-1, +1)$. Indeed, consider a subsystem $\mathcal{B}_{0,5}^3$ of B_{k+q}^3 containing the orbits \hat{O} and $\tilde{\omega}$ and such that, in any sequence (22),

- if k_s is odd, then k_{s+1} is even and such that $k_{s+1} < k_s$;
- if k_s is even, then either k_{s+1} is any even integer ($\geq \bar{k}$) or k_{s+1} is odd and such that $k_{s+1} > k_s$.

Proposition 3.12. *Let $f_0 \in H_3^5$ and $\tau \in (-1, +1)$. Then, for some $\bar{k} = \bar{k}(\tau) \rightarrow \infty$ as $|\tau| \rightarrow 1$, the system $f_0|_{N_{\bar{k}}}$ is topologically conjugate to $\mathcal{B}_{0,5}^3$.*

The proof follows immediately from the simple fact that inequality (16) automatically holds for the pointed out integers $j = k_s$ and $i = k_{s+1}$ when $\lambda = \gamma^{-1}$, $c < 0$, $d > 0$ and $|\tau| < 1$.

We see that the value $\tau = 0$ has a special meaning for the dynamics of APMs of the third class, only maps in H_3^5 are not sensitive to the global resonance. This feature will effectively come to light when we study bifurcations of single-round periodic orbits.

4 General unfoldings and bifurcations.

The main goal of the rest of the paper is the study of bifurcations of *single-round periodic orbits* in one and two parameter unfolding families f_ε of APMs (under condition **C**) with the initial quadratic homoclinic tangency of the map f_0 satisfying conditions **A** and **B**. Recall that, by Definition 1, every such an orbit has only one intersection point with Π^+ (or with Π^-). Thus, such a point can be considered as a fixed point of the corresponding *first return map* $T_k \equiv T_1 T_0^k : \sigma_k^0 \mapsto \sigma_k^0$ with an appropriate integer $k \geq \bar{k}$. Note that the integers k can run among all values in the set $\{\bar{k}, \bar{k} + 1, \dots\}$.

Concerning the parameter families we will consider either one parameter families with the parameter $\varepsilon = \mu$ (general case) or two parameter families with $\varepsilon = (\mu, \tau)$ (the global resonance case). The latter (two parameter) family will be used only to study the bifurcation of APMs $f_0 \in H_3^i$ with $i = 1, 2, 3, 4$, which are extremely sensitive to the resonance $\tau = 0$, see Section 3.4. Recall that μ is the parameter of splitting manifolds $W^u(O)$ and $W^s(O)$ with respect to the homoclinic point M^+ (see maps (14)), and τ is the invariant quantity given in (23).

4.1 The main rescaling lemma.

In principle, one can study bifurcations of the first return maps T_k written in the initial coordinates and with the initial parameters ε , using the corresponding formulae for the local map T_0 , its iterations T_0^k and the global map T_1 from Section 2. However, there is a more effective way for studying homoclinic bifurcations. Namely, we can bring maps T_k to some unified form for all large k using the so-called rescaling method as it has been done in many papers, see e.g. the papers [7, 1, 38, 30, 31] where the rescaling method was applied to the conservative and reversible cases. After this, we can study (once) bifurcations in the unified map and “project” the obtained results onto the first return maps T_k for various k .

The main technical result of this section is the following

Lemma 6. [The main rescaling lemma]

For every sufficiently large k the first return map $T_k : \sigma_k^0 \rightarrow \sigma_k^0$ can be brought, by a linear transformation of coordinates and parameters, to the following form

$$\begin{aligned}\bar{X} &= Y + k\lambda^{2k}\varepsilon_k^1, \\ \bar{Y} &= M - \nu_1 X - Y^2 + \nu_2 \frac{f_{03}}{d^2}\lambda^k Y^3 + k\lambda^{2k}\varepsilon_k^2,\end{aligned}\tag{32}$$

where $\nu_1 = \text{sign}(-bc\lambda^k\gamma^k)$, $\nu_2 = \text{sign}(\lambda^k\gamma^k)$; the functions $\varepsilon_k^{1,2}(X, Y, M)$ are defined on a ball $\|(X, Y, M)\| \leq R$ with arbitrary large R (when k are big) and are uniformly bounded in k along with all their derivatives up to order $(r - 4)$. Moreover, the following formulae take place for M depending on the saddle O being orientable or not: if $\lambda\gamma = +1$, then

$$M = -d(1 + \rho_k^1)\lambda^{-2k}(\mu + \lambda^k(cx^+ - y^-)(1 + k\beta_1\lambda^k x^+ y^-)) - s_0 + \rho_k^2,\tag{33}$$

if $\lambda\gamma = -1$, then

$$M = -d(1 + \rho_k^3)\lambda^{-2k} (\mu + c\lambda^k x^+ - \gamma^{-k} y^-) - s_0 + \rho_k^4, \quad (34)$$

where

$$s_0 = dx^+(ac + f_{20}x^+) + \frac{1}{2}f_{11}x^+ \left(1 + \nu_1 - \frac{1}{2}f_{11}x^+\right) \quad (35)$$

and $\rho_k^i = O(k\lambda^k)$ are some small coefficients.

Proof. We will use the representation of the map T_0 in the “second normal form”, i.e., as in (10).⁶ Then the map $T_0^k : \sigma_k^0 \rightarrow \sigma_k^1$, for all sufficiently large k , can be written in form (11).

First we consider the case where the saddle O is *orientable*, $\gamma^{-1} = \lambda$. Then, using (13), (14) and (11), we can write the first return map $T_k : \sigma_k^0 \rightarrow \sigma_k^0$ in the following form

$$\begin{aligned} \bar{x} - x^+ &= a\lambda^k x + b(y - y^-) + e_{02}(y - y^-)^2 + \\ &\quad + O(k|\lambda|^{2k}|x| + |y - y^-|^3 + |\lambda|^k|x||y - y^-|), \\ \lambda^k \bar{y} (1 + k\lambda^k \beta_1 \bar{x} \bar{y}) + k\lambda^{3k} O(|\bar{x}| + |\bar{y}|) &= \\ &= \mu + c\lambda^k x (1 + k\lambda^k \beta_1 xy) + d(y - y^-)^2 + \lambda^{2k} f_{20} x^2 + \\ &\quad + \lambda^k f_{11} (1 + k\lambda^k \beta_1 xy) x(y - y^-) + \lambda^k f_{12} x(y - y^-)^2 + f_{03}(y - y^-)^3 + \\ &\quad + O((y - y^-)^4 + \lambda^{2k}|x||y - y^-| + k|\lambda|^{3k}|x| + k\lambda^{2k}|x||y - y^-|^2), \end{aligned} \quad (36)$$

where we use the Shilnikov cross-coordinates $x = x_0, y = y_k$, which are very convenient for the construction of return maps near saddles (see e.g. [11]).

Below, we will denote by $\alpha_{ki} = O(k\lambda^k)$, $i = 0, 1, 2, \dots$, some asymptotically small coefficients. Now we shift the coordinates

$$\eta = y - y^-, \quad \xi = x - x^+ - \lambda^k x^+(a + \alpha_k^0),$$

in order to vanish the constant term (independent of coordinates) in the first equation of (36). Thus, (36) is recast as follows

$$\begin{aligned} \bar{\xi} &= a\lambda^k \xi + b(1 + \alpha_{k1})\eta + e_{02}\eta^2 + O(k\lambda^{2k}|\xi| + |\eta|^3 + |\lambda|^k|\xi||\eta|), \\ \lambda^k \bar{\eta} (1 + \alpha_{k2}) + k\lambda^{2k} O(|\bar{\xi}| + |\bar{\eta}|^2) + k\lambda^{3k} O(|\bar{\eta}|) &= M_1 + c\lambda^k \xi (1 + \alpha_{k3}) + \\ &\quad + \eta^2 (d + \lambda^k f_{12} x^+) + \lambda^k \eta (f_{11} x^+ + \alpha_{k4}) + \lambda^k f_{11} \xi \eta + f_{03} \eta^3 + \\ &\quad + O(\eta^4 + k|\lambda|^{3k}|\xi| + k\lambda^{2k}(\xi^2 + \eta^2) + \lambda^k |\xi| \eta^2), \end{aligned} \quad (37)$$

where

$$M_1 = \mu + \lambda^k (cx^+ - y^-) (1 + k\lambda^k \beta_1 x^+ y^-) + \lambda^{2k} x^+ (ac + f_{20} x^+) + O(k\lambda^{3k}). \quad (38)$$

Now we rescale the variables:

$$\xi = -\frac{b(1 + \alpha_{k1})(1 + \alpha_{k2})}{d + \lambda^k f_{12} x^+} \lambda^k u, \quad \eta = -\frac{1 + \alpha_{k2}}{d + \lambda^k f_{12} x^+} \lambda^k v. \quad (39)$$

⁶Of course we lose a little in the smoothness, since the second order normal form is C^{r-2} only, see Lemma 2. However, we gain important information on form of the first return maps. On the other hand, our considerations cover also the C^∞ and real analytical cases.

System (37) in the coordinates (u, v) is rewritten in the following form

$$\begin{aligned}\bar{u} &= v + a\lambda^k u - \frac{e_{02}}{bd}\lambda^k v^2 + O(k\lambda^{2k}), \\ \bar{v} &= M_2 - \nu_1 u(1 + \alpha_{k5}) - v^2 + \\ &\quad + v(f_{11}x^+ + \alpha_{k6}) - \frac{f_{11}b}{d}\lambda^k uv + \frac{f_{03}}{d^2}\lambda^k v^3 + O(k\lambda^{2k}),\end{aligned}\tag{40}$$

where $\nu_1 = -bc$ since the saddle O is orientable for the case under consideration, i.e., $\nu_1 = 1$, if T_1 (and also T_k) is orientable map, and $\nu_1 = -1$ if T_1 is non-orientable (the globally non-orientable case), and

$$M_2 = -\frac{d + \lambda^k f_{12}x^+}{1 + \alpha_{k2}}\lambda^{-2k}M_1.$$

By the following shift of the coordinates

$$u_{new} = u - \frac{1}{2}(f_{11}x^+ + \alpha_{k6}), \quad v_{new} = v - \frac{1}{2}(f_{11}x^+ + \alpha_{k7})\tag{41}$$

with $\alpha_{k6}, \alpha_{k7} = O(\lambda^k)$, we bring the map (40) into the following form

$$\begin{aligned}\bar{u} &= v + a\lambda^k u - \frac{e_{02}}{bd}\lambda^k v^2 + O(k\lambda^{2k}), \\ \bar{v} &= M_3 - \nu_1 u - v^2 - \frac{f_{11}b}{d}\lambda^k uv + \frac{f_{03}}{d^2}\lambda^k v^3 + O(k\lambda^{2k}),\end{aligned}\tag{42}$$

where

$$M_3 = M_2 - \frac{f_{11}x^+}{2}(1 + \nu_1) + \frac{(f_{11}x^+)^2}{4}.$$

Now we make the following linear change of coordinates

$$x = u + \tilde{\nu}_k^1 v, \quad y = v + \tilde{\nu}_k^2 u,\tag{43}$$

where

$$\tilde{\nu}_k^1 = \frac{e_{02}}{bd}\lambda^k, \quad \tilde{\nu}_k^2 = a\lambda^k - \nu_1 \frac{e_{02}}{bd}\lambda^k.\tag{44}$$

Then system (42) is rewritten as

$$\begin{aligned}\bar{x} &= y + M_3\tilde{\nu}_k^1 + O(k\lambda^{2k}), \\ \bar{y} &= M_3 - \nu_1 x - y^2 + a\lambda^k y - \tilde{R}\lambda^k xy + \frac{f_{03}}{d^2}\lambda^k y^3 + O(k\lambda^{2k}),\end{aligned}\tag{45}$$

where $\tilde{R} = (2a - 2e_{02}\nu_1/bd - bf_{11}/d)$. Since $\nu_1 = -bc$, we obtain, by (15), that $\tilde{R} = \frac{1}{d}(2ad - 2ce_{02} - bf_{11}) \equiv 0$. Thus, the map (45) takes the following form

$$\begin{aligned}\bar{x} &= y + M_3\tilde{\nu}_k^1 + O(k\lambda^{2k}), \\ \bar{y} &= M_3 - \nu_1 x - y^2 + a\lambda^k y + \frac{f_{03}}{d^2}\lambda^k y^3 + O(k\lambda^{2k}).\end{aligned}\tag{46}$$

Finally, we make the last shift of the coordinates:

$$X = x - \frac{1}{2}a\lambda^k - \tilde{\nu}_k^1 M_3, \quad Y = y - \frac{1}{2}a\lambda^k,\tag{47}$$

in order to cancel the constant term in the first equation and the linear term in y in the second equation of (46). After this, we obtain the final form (32) of the map T_k in the rescaled coordinates, where formula (33) takes place for the parameter M .

The proof in the case of the *non-orientable saddle* O , i.e., when $\gamma^{-1} = -\lambda$, is quite similar and it is completely the same when k is even, taking into account that $\beta_1 = 0$. If k is odd, then $\gamma^{-k} = -\lambda^k$ and the corresponding formulae change. Therefore, we consider the case of odd k . By (12), system (37) can be written as follows

$$\begin{aligned}\bar{\xi} &= a\lambda^k\xi + (b + \hat{\alpha}_{k1})\eta + e_{02}\eta^2 + O(\lambda^{2k}(|\xi| + |\eta|) + |\eta|^3), \\ \bar{\eta} &= -M_1\lambda^{-k} - (c + \hat{\alpha}_{k2})\xi - (d + \lambda^k f_{12}x^+)\lambda^{-k}\eta^2 - (f_{11}x^+ + \hat{\alpha}_{k3})\eta - f_{11}\xi\eta - f_{03}\lambda^{-k}\eta^3 + \\ &\quad + O(|\lambda|^{-k}\eta^4 + |\eta|^3 + |\lambda|^k(\xi^2 + \eta^2)),\end{aligned}\quad (48)$$

where (recall that $\beta_1 = 0$ now)

$$M_1 = \mu + c\lambda^k x^+ + \lambda^k y^- + \lambda^{2k} x^+ (ac + f_{02}x^+) + O(\lambda^{3k}) \quad (49)$$

and $\hat{\alpha}_{ki} = O(\lambda^k)$, $i = 1, 2, \dots$, are some small coefficients.

After the rescaling

$$\xi = \frac{(b + \hat{\alpha}_{k1})\lambda^k}{d + \lambda^k f_{12}x^+} u, \quad \eta = \frac{\lambda^k}{d + \lambda^k f_{12}x^+} v \quad (50)$$

system (48) is rewritten in the following form

$$\begin{aligned}\bar{u} &= v + a\lambda^k u + \frac{e_{02}}{bd}\lambda^k v^2 + O(\lambda^{2k}), \\ \bar{v} &= M_2 + u(1 + \hat{\alpha}_{k4}) - v^2 - (f_{11}x^+ + \hat{\alpha}_{k3})v - \frac{f_{11}b}{d}\lambda^k uv - \frac{f_{03}}{d^2}\lambda^k v^3 + O(\lambda^{2k}),\end{aligned}\quad (51)$$

where $M_2 = -(d + \lambda^k f_{12}x^+)\lambda^{-2k}M_1$.

Recall that in this (locally non-orientable) case, by condition **D**, the homoclinic points M^+ and M^- are of the needed type, i.e., the global map T_1 is orientable: $bc = -1$. Then the first return map T_k for odd k will be non-orientable, i.e., the Jacobian of the map (51) is equal identically to -1 .

After the coordinate shift (41) with appropriate $\alpha_{k6}, \alpha_{k7} = O(\lambda^k)$, the map (51) is recast as

$$\begin{aligned}\bar{u} &= v + a\lambda^k u + \frac{e_{02}}{bd}\lambda^k v^2 + O(\lambda^{2k}), \\ \bar{v} &= M_3 + u - v^2 - \frac{f_{11}b}{d}\lambda^k uv - \frac{f_{03}}{d^2}\lambda^k v^3 + O(\lambda^{2k}),\end{aligned}\quad (52)$$

where $M_3 = M_2 + \frac{1}{4}(f_{11}x^+)^2 + O(\lambda^k)$. After the linear change of coordinates (43)–(44) with $\nu_1 = -1$, the system (52) takes the form

$$\bar{x} = y + M_3\tilde{\nu}_k^1 + O(\lambda^{2k}), \quad \bar{y} = M_3 + x - y^2 + a\lambda^k y - \frac{f_{03}}{d^2}\lambda^k y^3 + O(\lambda^{2k}). \quad (53)$$

Note that in the second equation of (53), the same as in (46), the term with xy vanishes. Note also that the sign before y^3 is opposite to that in (46).

Finally, by means of the coordinate shift (47), we bring the map (53) into the form (32). This completes the proof. \square

4.2 On bifurcations of fixed points in the conservative Hénon maps.

The Rescaling Lemma 6 shows that the limit rescaled form of the first return maps T_k is the conservative Hénon map which is orientable if $\nu_1 = 1$, and non-orientable if $\nu_1 = -1$ (recall that ν_1 is the Jacobian of T_k , i.e., $\nu_1 = 1$ if T_k is orientable and $\nu_1 = -1$ if T_k is non-orientable). Bifurcations of fixed points in these conservative maps are well-known.

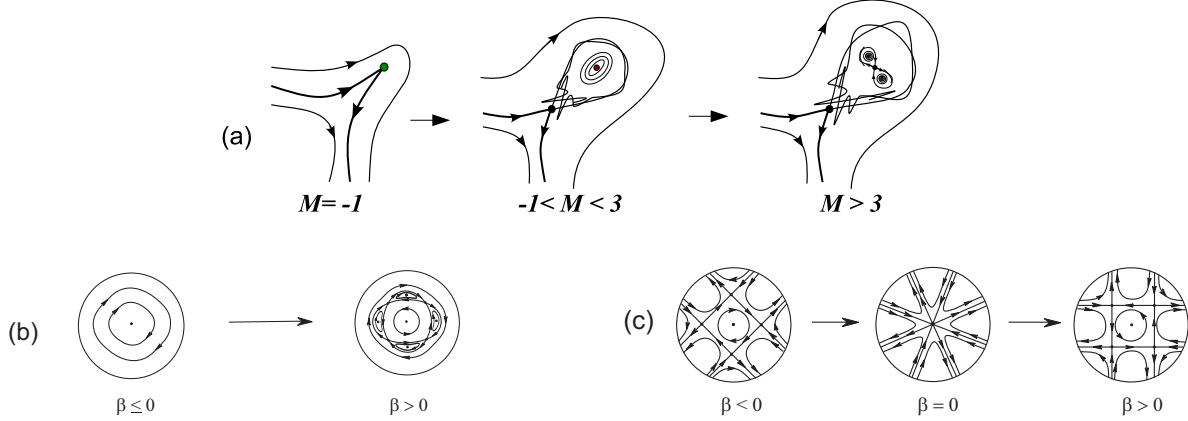


Figure 12: Bifurcations of fixed points in the orientable conservative Hénon map: (a) the main scenario: $M < -1$, there are no fixed points; $M = -1$, a fixed parabolic point appears; $-1 < M < 3$, two saddle and elliptic fixed points exist; $M = -3$, the period doubling bifurcation with the elliptic fixed point; (b)–(c) bifurcations near resonance $1 : 4$ in the rescaled first return map (32) for the cases (b) $\hat{\nu}_k = \nu_2 f_{03} d^{-2} \lambda^k > 0$ (here the fixed point is always elliptic with multipliers $e^{\pm i\psi}$) and (c) $\hat{\nu}_k < 0$ (at $\beta = 0$ the fixed point is a saddle with eight separatrices) – here β is a parameter characterizing a deviation of ψ from $\pi/2$.

4.2.1 The orientable case.

In the orientable case $\nu_1 = 1$, the conservative Hénon map

$$\bar{x} = y, \quad \bar{y} = M - x - y^2 \quad (54)$$

has an elliptic fixed point with multipliers $e^{\pm i\psi}$, $\psi = \arccos(1 - \sqrt{1 + M})$, for $M \in (-1, 3)$. This point is generic (KAM-stable) for all $M \in (-1, 3)$ except for two values: $M = 0$ for $\psi = \pi/2$ and $M = 5/4$ for $\psi = 2\pi/3$. The fixed point for $\psi = 2\pi/3$ is non-degenerate and always unstable: it is a saddle with six separatrices. On the other hand, the fixed point for $\psi = \pi/2$ is degenerate: the so-called case “ $A = 1$ ”, [53, 54], takes place in the Hénon map. However, the map T_k , in reality, has the rescaled form (32). Therefore, if the coefficient f_{03} is non-zero, the conservative resonance $1 : 4$ becomes non-degenerate [6, 25]. Namely, the corresponding fixed point will be KAM-stable (of elliptic type) if $f_{03} \lambda^k > 0$ and unstable (a saddle with 8 separatrices) if $f_{03} \lambda^k < 0$, see Figure 12(b),(c).

The conservative Hénon map has also fixed parabolic points, for $M = -1$ with double multiplier $+1$, and for $M = 3$ with double multiplier -1 . The corresponding conservative bifurcations are non-degenerate. See Figure 12 for an illustration.

4.2.2 The non-orientable case.

In the non-orientable case $\nu_1 = -1$, the conservative non-orientable Hénon map

$$\bar{x} = y, \quad \bar{y} = M + x - y^2 \quad (55)$$

does not have elliptic fixed points. However, elliptic 2-periodic orbits exist here for $M \in (0, 1)$.

The non-orientable Hénon map (55) has no fixed points for $M < 0$, it has one fixed point $\bar{O}(0, 0)$ with multipliers $\nu_1 = +1, \nu_2 = -1$ for $M = 0$ and two saddle fixed points, $\bar{O}_1(-\sqrt{M}, -\sqrt{M})$ and $\bar{O}_2(\sqrt{M}, \sqrt{M})$, for $M > 0$. Besides, an elliptic 2-periodic orbit exists for $0 < M < 1$, consisting of two points $p_1(-\sqrt{M}, \sqrt{M})$ and $p_2(\sqrt{M}, -\sqrt{M})$ and has multipliers $e^{\pm i\psi}$, where $\psi = \arccos(1 - 2M)$. The value $M = +1$ corresponds to the period doubling bifurcation of this elliptic orbit. See Figure 13 for an illustration.

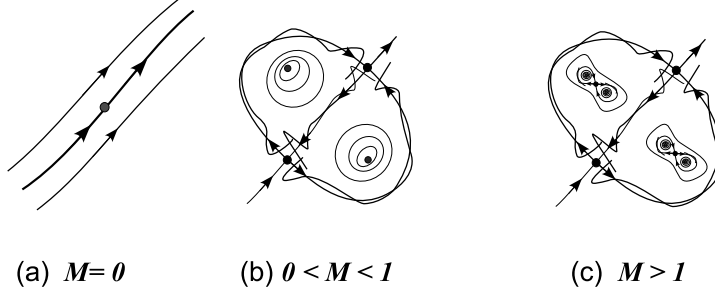


Figure 13: The main bifurcation scenario in the non-orientable conservative Hénon map.

Note that the elliptic 2-periodic orbit is generic for all $M \in (0, 1)$, except for $M = 1/2$ and $M = 3/4$ which correspond to the strong resonances $1 : 4$ and $1 : 3$, respectively, and $M = 5/8$ which corresponds to the cancellation of the first Birkhoff coefficient at the cycle $\{p_1, p_2\}$, see [31].⁷

5 One parameter cascades of elliptic periodic orbits.

In this section we consider the problem on existence of cascades of elliptic periodic orbits in the case of APMs with quadratic homoclinic tangencies. We establish not only their existence but we also analyze questions on the coexistence of elliptic periodic orbits of different periods. This will allow us to construct the main elements of the bifurcation diagrams near homoclinic tangencies for APMs.

Let f_0 be an APM with a quadratic homoclinic tangency for which conditions **A** and **B** hold. We embed f_0 in a one parameter family f_μ of APMs which unfolds generally, under condition **C**, the initial quadratic homoclinic tangency.

1. We first consider *the symplectic case*.

By the Rescaling Lemma 6, the conservative (orientable) Hénon map (54) is the limit rescaled form for the first return maps T_k with sufficiently large k . In this case, we can rewrite the relation (33) as follows

$$\mu = -\lambda^k y^- \alpha(1 + k\beta_1 \lambda^k x^+ y^-) - \frac{1}{d}(M + s_0)\lambda^{2k} + O(k\lambda^{3k}), \quad (56)$$

where

$$s_0 = dx^+(ac + f_{20}x^+) + f_{11}x^+ \left(1 - \frac{1}{4}f_{11}x^+\right) \quad (57)$$

⁷However, we do not study here the question of KAM-stability of the corresponding 2-periodic orbits with $\psi = \pi/2, 2\pi/3, \arccos -1/4$. Note that in the case of the conservative (and orientable) Hénon map the corresponding problem was solved in [6], where it was established that the fixed elliptic point of the Hénon map at $M = 9/16$, with $\psi = \arccos(-1/4)$, has zero first Birkhoff coefficients and nonzero second one, i.e., it is KAM-stable.

(that gives formula (35) with $\nu_1 = 1$) and

$$\alpha = \frac{cx^+}{y^-} - 1. \quad (58)$$

Since the parabolic fixed points exist in the map (54) for $M = -1$ and $M = 3$, we obtain that the first return map T_k , for sufficiently large k , has a fixed point with double multiplier $+1$ and with double multiplier -1 , respectively, for such values of μ :

$$\begin{aligned} \mu = \mu_k^+ &\equiv -\lambda^k y^- \alpha (1 + k\beta_1 \lambda^k x^+ y^-) - \frac{1}{d}(s_0 - 1)\lambda^{2k} + O(k\lambda^{3k}), \\ \mu = \mu_k^- &\equiv -\lambda^k y^- \alpha (1 + k\beta_1 \lambda^k x^+ y^-) - \frac{1}{d}(s_0 + 3)\lambda^{2k} + O(k\lambda^{3k}). \end{aligned} \quad (59)$$

Thus, the values $\mu = \mu_k^+$ and $\mu = \mu_k^-$ correspond to the border points of the interval \mathbf{e}_k such that the first return map T_k has an elliptic fixed point for $\mu \in \mathbf{e}_k$. These elliptic points are generic (KAM-stable) for all values of $\mu \in \mathbf{e}_k$, except for the two values $\mu = \mu_k^{\pi/2}$ and $\mu = \mu_k^{2\pi/3}$ corresponding, respectively, to $M = 0$ and $M = 5/4$ in (56).

2. Second, we consider *the globally non-orientable case*.

In this case, f_0 satisfies conditions **A** and **B** with $\lambda\gamma = +1$ and also the global map T_1 is non-orientable, i.e., $bc = +1$. Then, by the Rescaling Lemma 6, the conservative *non-orientable* Hénon map (55) will be the limit rescaled form for the first return maps T_k with sufficiently large k . Note that formula (33), describing the relation between the parameters M and μ , takes place here as before, however, since $bc = +1$, the quantity s_0 will be different from the one of the symplectic case:

$$s_0 = s_0^{nor} = dx^+(ac + f_{20}x^+) - \frac{1}{4}(f_{11}x^+)^2 \quad (60)$$

(that gives formula (35) with $\nu_1 = -1$). Note that in this (non-orientable) case we use the notation s_0^{nor} for s_0 , only for more definiteness.

Then in the globally non-orientable case, the relation (33) is rewritten as follows

$$\mu = -\lambda^k y^- \alpha (1 + k\beta_1 \lambda^k x^+ y^-) - \frac{1}{d}(M + s_0^{nor})\lambda^{2k} + O(k\lambda^{3k}), \quad (61)$$

Since the map (55) has parabolic-like fixed points for $M = 0$ and $M = 1$, we obtain, by (61), that the first return map T_k , for sufficiently large k , has a fixed point with multipliers $+1$ and -1 and a 2-periodic orbit with double multiplier -1 , respectively, for such values of μ :

$$\begin{aligned} \mu = \mu_k^{\pm 1} &\equiv -\lambda^k y^- \alpha (1 + k\beta_1 \lambda^k x^+ y^-) - \frac{1}{d}s_0^{nor} \lambda^{2k} + O(k\lambda^{3k}), \\ \mu = \mu_k^{2-} &\equiv -\lambda^k y^- \alpha (1 + k\beta_1 \lambda^k x^+ y^-) - \frac{1}{d}(s_0^{nor} + 1)\lambda^{2k} + O(k\lambda^{3k}). \end{aligned} \quad (62)$$

Thus, the values $\mu = \mu_k^{\pm 1}$ and $\mu = \mu_k^{2-}$ correspond to the border points of the interval \mathbf{e}_k^2 such that the first return map T_k has an elliptic 2-periodic orbit for $\mu \in \mathbf{e}_k^2$. These elliptic 2-periodic orbits are generic (KAM-stable) for all values of $\mu \in \mathbf{e}_k^2$, except for three values $\mu = \mu_k^{2,\pi/2}$, $\mu = \mu_k^{2,2\pi/3}$ and $\mu = \mu_k^{2,\arccos -1/4}$ corresponding, respectively, to $M = 1/2$, $M = 3/4$ and $M = 5/8$ in (56).

3. Finally, we consider *the locally non-orientable case*.

In this case f_0 satisfies conditions **A** and **B** with $\lambda\gamma = -1$ and $bc = -1$ by condition **D**. In the case under consideration, $f_0 \in H_3^2 \cup H_3^3$, see section 3.4. Then the first return maps T_k will be orientable for even k and non-orientable for odd k .

First, we consider the case of *even* k , $k = 2m$.

Then, by the Rescaling Lemma 6, the conservative orientable Hénon map (54) is the limit rescaled form for the first return maps T_k , where M satisfies (34) and, the same as in the symplectic case, formula (57) holds for s_0 . We can rewrite relation (34) for even k as

$$\mu = -\lambda^k y^- \alpha - \frac{1}{d}(s_0 + M)\lambda^{2k} + O(k\lambda^{3k}), \quad k = 2m \quad (63)$$

and obtain that the first return map T_k , for sufficiently large *even* k , has a fixed point with double multiplier $+1$ and with double multiplier -1 , respectively, for such values of μ :

$$\begin{aligned} \mu = \tilde{\mu}_k^+ &\equiv -\lambda^k y^- \alpha - \frac{1}{d}(s_0 - 1)\lambda^{2k} + O(k\lambda^{3k}), \\ \mu = \tilde{\mu}_k^- &\equiv -\lambda^k y^- \alpha - \frac{1}{d}(s_0 + 3)\lambda^{2k} + O(k\lambda^{3k}), \quad k = 2m. \end{aligned} \quad (64)$$

Thus, the values $\mu = \tilde{\mu}_{2m}^+$ and $\mu = \tilde{\mu}_{2m}^-$ correspond to the border points of the interval $\tilde{\mathbf{e}}_{2m}$ such that the first return map T_{2m} has an elliptic fixed point for $\mu \in \tilde{\mathbf{e}}_{2m}$. These elliptic fixed points are generic (KAM-stable) for all values of $\mu \in \tilde{\mathbf{e}}_k$, except for the two values $\mu = \tilde{\mu}_k^{\pi/2}$ and $\mu = \tilde{\mu}_k^{2\pi/3}$ corresponding, respectively, to $M = 0$ and $M = 5/4$ in (63).

In the case where k is *odd*, $k = 2m + 1$, by the Rescaling Lemma 6, the conservative non-orientable Hénon map (55) becomes the limit rescaled form for the first return maps T_k , where M satisfies (34). However, for odd k , as in the globally non-orientable case, formula (60) holds for $s_0 = s_0^{nor}$. Thus, when k is odd, since $\lambda^k = -\gamma^{-k}$ and $\lambda < 0, \gamma > 0$ for $f_0 \in H_3^j$, $j = 2, 3$, relation (34) can be rewritten as

$$\mu = -\lambda^k y^- (\alpha + 2) - \frac{1}{d}(s_0^{nor} + M)\lambda^{2k} + O(k\lambda^{3k}), \quad k = 2m + 1 \quad (65)$$

where $\alpha + 2 = \frac{cx^+}{y^-} + 1$, see (58). Then we obtain that the first return map T_k , for sufficiently large *odd* k , has a fixed point with multipliers $+1$ and -1 and an elliptic 2-periodic orbit with double multiplier -1 , respectively, for such values of μ :

$$\begin{aligned} \mu = \tilde{\mu}_k^{\pm 1} &\equiv -\lambda^k y^- (\alpha + 2) - \frac{1}{d}s_0^{nor} \lambda^{2k} + O(k\lambda^{3k}), \\ \mu = \tilde{\mu}_k^{2-} &\equiv -\lambda^k y^- (\alpha + 2) - \frac{1}{d}(s_0^{nor} + 1)\lambda^{2k} + O(k\lambda^{3k}), \quad k = 2m + 1. \end{aligned} \quad (66)$$

Thus, the values $\mu = \tilde{\mu}_{2m+1}^{\pm 1}$ and $\mu = \tilde{\mu}_{2m+1}^{2-}$ correspond to the border points of the interval $\tilde{\mathbf{e}}_{2m+1}^2$ such that the first return map T_{2m+1} has an elliptic 2-periodic orbit for $\mu \in \tilde{\mathbf{e}}_{2m+1}^2$. These elliptic 2-periodic orbits are generic (KAM-stable) for all values of $\mu \in \tilde{\mathbf{e}}_k^2$, except for the three values $\mu = \tilde{\mu}_k^{2, \pi/2}$, $\mu = \tilde{\mu}_k^{2, 2\pi/3}$ and $\mu = \tilde{\mu}_k^{2, \arccos -1/4}$ corresponding, respectively, to $M = 1/2$, $M = 3/4$ and $M = 5/8$ in (65).

Now we collect the results obtained in this section in the following theorem.

Theorem 1. [On a one parameter cascade of elliptic points]

Let f_μ be a one parameter family of APMs satisfying conditions **A**, **B** and **C**. Then in any segment $[-\mu_0, \mu_0]$ of μ , there exist infinitely many intervals (i) \mathbf{e}_k , $k = \bar{k}, \bar{k} + 1, \dots$, in the symplectic case; (ii) \mathbf{e}_k^2 , $k = \bar{k}, \bar{k} + 1, \dots$, in the globally non-orientable case; (iii) $\tilde{\mathbf{e}}_{2m}$ and $\tilde{\mathbf{e}}_{2m+1}^2$, $m = \bar{m}, \bar{m} + 1, \dots$, in the locally non-orientable case⁸ such that

1.a) f_μ has a single-round elliptic periodic orbit (of period $k + q$) either at (i) $\mu \in \mathbf{e}_k$ or at (iii) $\mu \in \tilde{\mathbf{e}}_{2m}$, where $k = 2m$;

1.b) f_μ has a double-round elliptic periodic orbit (of period $2(k + q)$ corresponding to a 2-periodic point of T_k) either at (ii) $\mu \in \mathbf{e}_k^2$ or at (iii) $\mu \in \tilde{\mathbf{e}}_{2m+1}^2$, where $k = 2m + 1$.

2) These elliptic orbits are generic (KAM-stable) at almost all values of the parameter μ in the pointed out intervals (except for those values which correspond to the points with multipliers $e^{\pm i\psi}$, where $\psi = \pi/2, 2\pi/3$ in all cases and $\psi = \arccos(-1/4)$ in the cases (ii) as well as (iii) with $k = 2m + 1$).

3) If $\alpha \neq 0$ (α is given in (58)), the intervals \mathbf{e}_k and \mathbf{e}_k^2 as well as $\tilde{\mathbf{e}}_k$ with $k = 2m$ do not intersect for different large k ; if $\alpha \neq -2$, the intervals $\tilde{\mathbf{e}}_{2m+1}^2$ do not intersect for different large m .

For the symplectic case, Theorem 1 was established in [5]. Indeed, also in [1] the existence of cascade of elliptic periodic orbits was proved, however, the problem on their coexistence was not considered.

We notice that the cases of global resonances, that is, maps f_0 with $\alpha = 0$ as well as with $\alpha = -2$ for $f_0 \in H_3^{2,2}, H_3^{3,2}$, are of special interest, since elliptic periodic orbits (even infinitely many of them) can coexist. The related phenomena will be considered in next section.

6 On bifurcations of single-round periodic orbits in two parameter general unfoldings.

In principle, in Section 5 we have studied bifurcations of single-round periodic orbits for APMs with quadratic homoclinic tangencies. Only some small questions remain unanswered, e.g. bifurcations of strong resonances in the non-orientable cases etc. However, we have not yet constructed more or less complete bifurcation diagrams which include not only the results related to bifurcations of concrete (single-round) periodic orbits but also, what is more important, the problem of coexistence of (elliptic) periodic orbits and, correspondingly, the problem of the order of bifurcations. When the intervals of existence of elliptic orbits of different periods do not intersect, we can assume that the bifurcation problem has been solved completely (up to some small details). However, as we saw, in the case under consideration, these intervals (from Theorem 1) may intersect and, moreover, they intersect undoubtedly when values of the parameter α varies near zero (or near $\alpha = -2$ when $f_0 \in H_3^{j,2}, j = 2, 3$). Thus, the cases of the maps f_0 with $\alpha = 0$ (respectively, $\alpha = -2$) are special and it is clear that these cases require to consider at least two parameter families including the parameters μ and α as the governing ones.

Note that symplectic two-dimensional maps with quadratic homoclinic tangencies with $\alpha = 0$ were studied in the papers by Gonchenko and Shilnikov [2, 3], where the phenomenon of “global resonance” was discovered. This phenomenon consists in the fact that for the values $\mu = 0, \alpha = 0$ the map f_0 can have infinitely many single-round elliptic periodic orbits of *all successive periods* starting at some number. This unusual dynamical property takes place when $-3 < s_0 < 1$,

⁸ with border points given by formulae (i) (59); (ii) (62); (iii) (64) with $k = 2m$ and, respectively, (66) with $k = 2m + 1$

i.e., this is a codimension 2 effect. This phenomenon can be considered as very interesting from various points of view, since elliptic orbits play an important rôle in conservative dynamics and applications (including Celestial Mechanics, [55], smooth billiards, [56], etc).

In the present paper we also observe this phenomenon but from a different point of view: we study bifurcation diagrams (for single-round periodic orbits) in two parameter families, say $f_{\mu,\alpha}$, and show that all the domains of existence of single-round elliptic orbits can contain the point $\mu = 0, \alpha = 0$. Moreover, we show that the phenomenon of global resonance takes place also in the non-symplectic case.

Note that the invariants τ , see (23) and Section 3, and α are closely related:

$$\tau = \frac{1}{\ln|\lambda|} \ln|\alpha + 1| \Leftrightarrow \begin{cases} \alpha = |\lambda|^\tau - 1 & \text{if } \alpha > -1 \\ \alpha = -1 - |\lambda|^\tau & \text{if } \alpha < -1 \end{cases}$$

and, thus, both values $\alpha = 0$ and $\alpha = -2$ are equivalent to $\tau = 0$. However, α and τ appear in homoclinic dynamics in different ways: τ is a natural parameter describing the structure of nonwandering orbits of f_0 , i.e., when the tangency takes place, and α is a natural parameter when studying bifurcations within the family f_μ . However, in principle, they are the same. Therefore, in this section, we will study bifurcation by means of the families $f_{\mu,\alpha}$ which unfold generally the initial tangency at $\mu = 0, \alpha = 0$, except for the case $f_0 \in H_3^{j,2}, j = 2, 3$ where the initial tangency exists at $\mu = 0, \alpha = -2$. We do this more for the simplicity of the presentation, since, in fact, we have already got, in Section 5, the formulae (namely, (59), (62), (64) and (66)) for the main bifurcation curves on the plane of parameters μ and α . We introduce the following notations for the bifurcation curves and the corresponding domains of existence of elliptic periodic orbits.

Definition 3. • For the *symplectic case*, we denote the curves (59) by B_k^+ and B_k^- , i.e., for $(\mu, \alpha) \in B_k^+$ (respectively for $(\mu, \alpha) \in B_k^-$), the map $f_{\mu,\alpha}$ has a single-round periodic (of period $(k + q)$) orbit with double multiplier $+1$ (respectively with double multiplier -1). Denote also by E_k the domain between the curves B_k^+ and B_k^- , where the map $f_{\mu,\alpha}$ has a single-round elliptic periodic orbit (of period $(k + q)$).

- For the *globally non-orientable case*, we denote the curves (62) by $B_k^{\pm 1}$ and B_k^{2-} , i.e., the map $f_{\mu,\alpha}$ has a single-round periodic (of period $(k + q)$) orbit with multipliers 1 and -1 for $(\mu, \alpha) \in B_k^{\pm 1}$ and a double-round (of period $2(k + q)$) periodic orbit with double multiplier -1 for $(\mu, \alpha) \in B_k^{2-}$. Denote also by $E_k^{\pm 1}$ the domain between the curves $B_k^{\pm 1}$ and B_k^{2-} , where the map $f_{\mu,\alpha}$ has a double-round elliptic periodic orbit (corresponding to an elliptic 2-periodic orbit of the first return map T_k).
- For the *locally non-orientable case* we use the following notations. Denote the curves (64) with even $k = 2m$ by \tilde{B}_{2m}^+ and \tilde{B}_{2m}^- , i.e., for $(\mu, \alpha) \in \tilde{B}_{2m}^+$ (respectively for $(\mu, \alpha) \in \tilde{B}_{2m}^-$), the map $f_{\mu,\alpha}$ has a single-round periodic (of period $(2m + q)$) orbit with double multiplier $+1$ (respectively with double multiplier -1). Denote also by \tilde{E}_{2m} the domain between the curves \tilde{B}_{2m}^+ and \tilde{B}_{2m}^- , where the map $f_{\mu,\alpha}$ has a single-round elliptic periodic orbit. We denote the curves (66) with $k = 2m + 1$ by $\tilde{B}_{2m+1}^{\pm 1}$ and \tilde{B}_{2m+1}^{2-} , i.e., the map $f_{\mu,\alpha}$ has a single-round periodic (of period $(2m + 1 + q)$) orbit with multipliers $+1$ and -1 for $(\mu, \alpha) \in \tilde{B}_{2m+1}^{\pm 1}$ and a double-round (of period $2(2m + 1 + q)$) periodic orbit with double multiplier -1 for $(\mu, \alpha) \in \tilde{B}_{2m+1}^{2-}$. Denote also by $\tilde{E}_{2m+1}^{\pm 1}$ the domain between the curves $\tilde{B}_{2m+1}^{\pm 1}$ and \tilde{B}_{2m+1}^{2-} , where the map $f_{\mu,\alpha}$ has a double-round elliptic periodic orbit (corresponding to an elliptic 2-periodic orbit of the first return map T_{2m+1}).

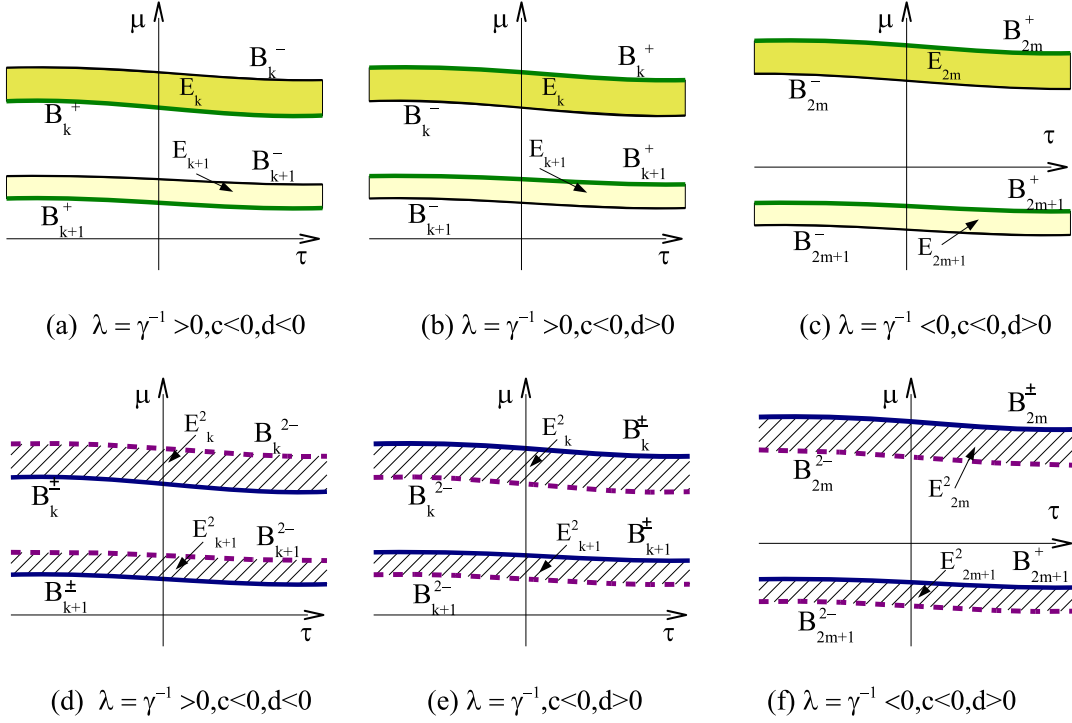


Figure 14: Elements of the bifurcation diagrams for families $f_{\mu,\tau}$ with $c < 0$: a)–c) the symplectic case; d)–f) the globally non-orientable case.

We note that the bifurcation diagrams in all cases with $c < 0$, except for the locally non-orientable case $f_0 \in H_3^{j,2}, j = 2, 3$, are simple, since, by Theorem 1, the intervals e_k (and e_k^2) of existence of single-round (double-round) elliptic periodic orbits do not intersect. If we now consider the family $f_{\mu,\tau}$, then we obtain a picture such as the one in Figure 14. More precisely, the following result is valid.

Proposition 6.1. *Let f_0 be a map of the first or second class or of the third class in H_3^5 and assume $\tau = 0$ in (23). Then, in any sufficiently small neighbourhood of the origin of the plane (μ, τ) , the domains E_k in the symplectic case or the domains E_k^2 in the globally non-orientable case do not intersect and accumulate as $k \rightarrow \infty$ to the axis $\mu = 0$, from one side for $\lambda = \gamma^{-1} > 0$ and from both sides for $\lambda = \gamma^{-1} < 0$.*

Proof. Since $c < 0$ in the cases under consideration, we have that $cx^+ - y^- = -y^- (|c|x^+/y^- + 1) = -y^- (|\lambda|^\tau + 1)$, by (23) and (58). Then we obtain from (59) the following equations for the curves B_k^+ and B_k^- :

$$\begin{aligned} B_k^+ : \quad \mu &= \lambda^k y^- (|\lambda|^\tau + 1) (1 + k\beta_1 \lambda^k x^+ y^-) + \frac{1 - s_0 + \dots}{d} \lambda^{2k}, \\ B_k^- : \quad \mu &= \lambda^k y^- (|\lambda|^\tau + 1) (1 + k\beta_1 \lambda^k x^+ y^-) - \frac{3 + s_0 + \dots}{d} \lambda^{2k}, \end{aligned} \quad (67)$$

which take place in the symplectic case. Accordingly, we obtain from (62) the following equations

for the curves B_k^- and B_k^{2-} :

$$\begin{aligned} B_k^- : \quad \mu &= \lambda^k y^- (|\lambda|^\tau + 1)(1 + k\beta_1 \lambda^k x^+ y^-) - \frac{1}{d}(s_0^{nor} + \hat{\rho}_k) \lambda^{2k}, \\ B_k^{2-} : \quad \mu &= \lambda^k y^- (|\lambda|^\tau + 1)(1 + k\beta_1 \lambda^k x^+ y^-) - \frac{1}{d}(s_0^{nor} + 1 + \hat{\rho}_k) \lambda^{2k}, \end{aligned} \quad (68)$$

which take place in the globally non-orientable case. The proposition follows immediately from these formulae, since $|\lambda|^\tau + 1 > 1$ and the “strips” E_k and E_k^2 have thickness of order λ^{2k} . \square

In the remaining cases of APMs with quadratic homoclinic tangencies, the structure of the domains of existence of elliptic periodic orbits near the origin ($\mu = 0, \tau = 0$) will be absolutely different, since the domains E_k , E_k^2 , \tilde{E}_k or \tilde{E}_k^2 can intersect. Moreover, an infinity of such domains can contain the origin and, hence, infinitely many single-round or double-round elliptic periodic orbits can coexist. Below we consider such phenomena.

First of all, we consider the symplectic case for which we establish the following result.

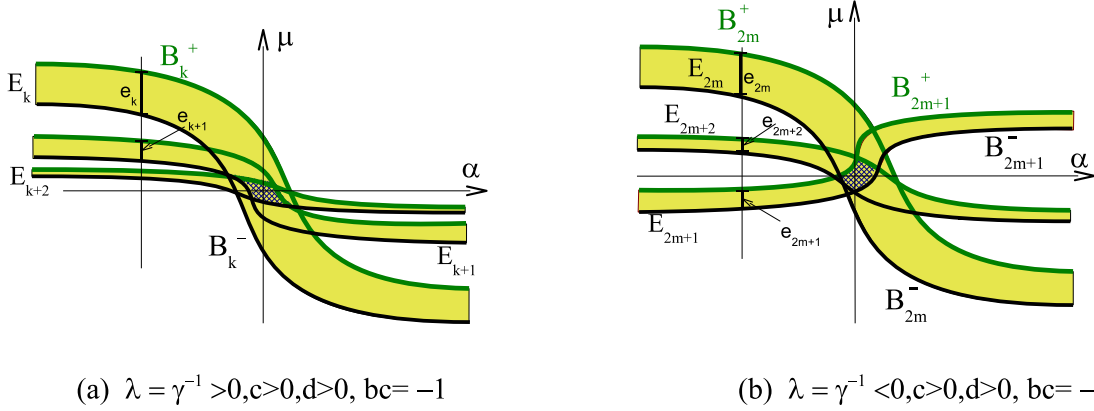


Figure 15: Elements of the bifurcation diagrams for families $f_{\mu,\alpha}$ in the case of symplectic maps f_0 of the third class in (a) H_3^1 and (b) H_3^4 . Here the case of “global resonance” at $\mu = 0, \alpha = 0$ is shown, when all the domains E_k contains the origin ($\mu = 0, \alpha = 0$).

Theorem 2. *Let $f_0 \in H_3^1 \cup H_3^4$ be a symplectic map and $f_{\mu,\alpha}$ be a two parameter general unfolding with the governing parameters μ and α . Then, in any sufficiently small neighbourhood of the origin of the parameter plane (μ, α) , there are infinitely many domains E_k , $k = \bar{k}, \bar{k} + 1, \dots$, which accumulate to the axis $\mu = 0$ as $k \rightarrow \infty$ and such that*

- 1) *all the domains E_k are mutually crossed and intersect the axis $\mu = 0$;*
- 2) *if $-3 < s_0 < 1$, all the domains E_k contains the origin ($\mu = 0, \alpha = 0$) and, hence, the map $f_{0,0}$ has infinitely many single-round elliptic periodic orbits of all successive periods $k = \bar{k} + q, \bar{k} + 1 + q, \dots$. Moreover, if $s_0 \neq 0, -5/4$ all these orbits are generic.*

Proof. In the symplectic case, when f_0 belongs to H_3^1 (where $\lambda = \gamma^{-1} > 0, c > 0, d > 0$) or H_3^4 (where $\lambda = \gamma^{-1} < 0, c > 0, d > 0$), the bifurcation curves B_k^+, B_k^- (the boundaries of the domain E_k) on the parameter plane (μ, α) are given by (59). It is easy to see from these formulae that all the curves B_k^+, B_k^- mutually intersect and that they cross the axis $\alpha = 0$ at $\mu = \mu_k^+ = -(d)^{-1}(s_0 - 1 + \dots)\lambda^{2k}$ and $\mu = \mu_k^- = -(d)^{-1}(s_0 + 3 + \dots)\lambda^{2k}$, and the axis $\mu = 0$ at the points

$\alpha = \alpha_k^+ = (dy^-)^{-1}(s_0 - 1 + \dots)\lambda^k$ and $\alpha = \alpha_k^- = (dy^-)^{-1}(s_0 + 3 + \dots)\lambda^k$. Then if $-3 < s_0 < 1$, all the domains E_k with sufficiently large k contains the origin ($\mu = 0, \alpha = 0$).

Moreover, by the Rescaling Lemma 6, all the first return maps T_k have “the same expression” for $\mu = 0, \alpha = 0$. Indeed, we obtain from (33) that the rescaled form (32) of T_k in this case looks as

$$\bar{X} = Y + O(k\lambda^{2k}), \bar{Y} = (-s_0 + \rho_k^2) - X + \frac{f_{03}}{d^2}\lambda^k Y^3 + O(k\lambda^{2k}), \quad (69)$$

where $\rho_k^2 = O(k\lambda^k)$ is a small coefficient (a correction to s_0). Then, see Section 4.2.1, if $-3 < s_0 < 1$ every map (69) with sufficiently large k has an elliptic fixed point which is generic if $s_0 \neq 0, -5/4$, i.e., if the strong resonances ($\psi = \pi/2, 2\pi/3$) are absent. \square

In Figure 15 we give an illustration of this theorem for different cases.

A similar result takes place in the globally non-orientable case.

Theorem 3. *Let $f_0 \in H_3^1 \cup H_3^4$ in the globally non-orientable case and $f_{\mu,\alpha}$ be a two parameter general unfolding of f_0 with the governing parameters μ and α . Then, in any sufficiently small neighbourhood of the origin of the parameter plane (μ, α) , there are infinitely many domains E_k^2 , $k = \bar{k}, \bar{k} + 1, \dots$, which accumulate to the axis $\mu = 0$ as $k \rightarrow \infty$ and such that 1) all the domains E_k^2 are mutually crossed and intersect the axis $\mu = 0$;*

2) *if $-1 < s_0^{nor} < 0$, all the domains E_k^2 contain the origin ($\mu = 0, \alpha = 0$) and, hence, the map $f_{0,0}$ has infinitely many double-round elliptic periodic orbits of all successive even periods $2(\bar{k} + q), 2(\bar{k} + q + 1), \dots$. Moreover, if $s_0 \neq -1/2, -3/4, -5/8$, all these orbits are generic.*

Proof. In the globally non-orientable case, when f_0 belongs to H_3^1 (with $\lambda = \gamma^{-1} > 0, c > 0, d > 0, bc = +1$) or H_3^4 (with $\lambda = \gamma^{-1} < 0, c > 0, d > 0, bc = +1$), the bifurcation curves $B_k^{\pm 1}, B_k^{2-}$ on the parameter plane (μ, α) are given by (62). It is easy to see from these formulae that all the curves $B_k^{\pm 1}, B_k^{2-}$ mutually intersect and cross the axis $\alpha = 0$ at the points $\mu = -(d)^{-1}(s_0^{nor} + \dots)\lambda^{2k}$ and $\mu = -(d)^{-1}(s_0^{nor} + 1 + \dots)\lambda^{2k}$, and the axis $\mu = 0$ at the points $\alpha = (dy^-)^{-1}(s_0^{nor} + \dots)\lambda^k$ and $\alpha = (dy^-)^{-1}(s_0^{nor} + 1 + \dots)\lambda^k$. Then, if $-1 < s_0 < 0$, all the domains E_k^2 with sufficiently large k contain the origin ($\mu = 0, \alpha = 0$).

Moreover, by the Rescaling Lemma 6, all the first return maps T_k have “the same expression” for $\mu = 0$ and $\alpha = 0$. Indeed, we obtain from (33) and (35) that the rescaled form (32) of T_k in this case looks as

$$\bar{X} = Y + O(k\lambda^{2k}), \bar{Y} = (-s_0^{nor} + \rho_k^4) + X + \frac{f_{03}}{d^2}\lambda^k Y^3 + O(k\lambda^{2k}). \quad (70)$$

Then, if $-1 < s_0 < 0$, every map (70) with sufficiently large k has an elliptic 2-periodic orbit which is generic if $s_0 \neq -1/2, -3/4, -5/8$, see Section 4.2.2. \square

In Figure 16 we give an illustration of this theorem.

We consider now the locally non-orientable case. Then we recall that $f_0 \in H_3^2 \cup H_3^3$, $\lambda\gamma = -1$ and $bc = -1$ (i.e., the local map T_0 is non-orientable and the global map T_1 is orientable).

Theorem 4. (I) *Let f_0 belong to $H_3^{2,1}$ or $H_3^{3,1}$ and $f_{\mu,\alpha}$ be a two parameter general unfolding with the governing parameters μ and α . Then in any sufficiently small neighbourhood V of the point $(\mu = 0, \alpha = 0)$ there are infinitely many domains \tilde{E}_{2m}^2 and \tilde{E}_{2m+1}^2 , $m = \bar{m}, \bar{m} + 1, \dots$, which accumulate to the axis $\mu = 0$ as $m \rightarrow \infty$, and the following holds.*

- Ia. *In V all the domains \tilde{E}_{2m}^2 are crossed and intersect the axis μ , whereas the domains \tilde{E}_{2m+1}^2 are not mutually crossed and do not intersect the axis μ .*

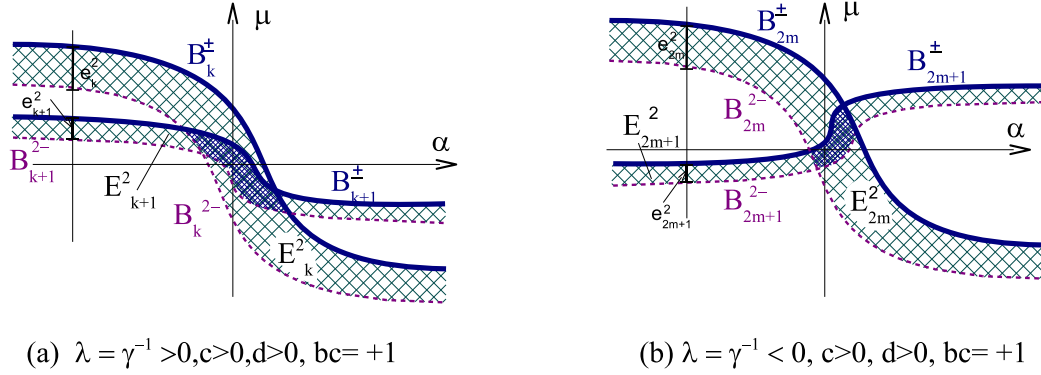


Figure 16: Elements of the bifurcation diagrams for families $f_{\mu,\alpha}$ in the globally non-orientable maps f_0 of the third class in (a) H_3^1 and (b) H_3^4 .

Ib. If $-3 < s_0 < 1$, all the domains \tilde{E}_{2m} contain the origin ($\mu = 0, \alpha = 0$) and, hence, the map $f_{0,0}$ has infinitely many single-round elliptic periodic orbits of all periods of the form $2m + q$, where $m = \bar{m}, \bar{m} + 1, \dots$. Moreover, if $s_0 \neq 0, -5/4$, all these orbits are generic.

(II) Let f_0 belong to $H_3^{2,2}$ or $H_3^{3,2}$ and $f_{\mu,\tilde{\alpha}}$ be a two parameter general unfolding with the governing parameters μ and $\tilde{\alpha}$.⁹ Then in any sufficiently small neighbourhood \tilde{V} of the origin ($\mu = 0, \tilde{\alpha} = 0$) there are infinitely many the domains \tilde{E}_{2m} and \tilde{E}_{2m+1}^2 , $m = \bar{m}, \bar{m} + 1, \dots$, which accumulate to the axis $\mu = 0$ as $m \rightarrow \infty$, and the following holds.

IIa. In \tilde{V} all the domains \tilde{E}_{2m+1}^2 mutually intersect, whereas the domains \tilde{E}_{2m} with different m do not intersect and do not cross the axis $\mu = 0$.

IIb. If $-1 < s_0^{nor} < 0$, all the domains \tilde{E}_{2m+1}^2 contain the point $(\mu = 0, \tilde{\alpha} = 0)$ and, hence, the map $f_{0,0}$ has infinitely many double-round elliptic periodic orbits of all periods of the form $2(2m + 1 + q)$, where $m = \bar{m}, \bar{m} + 1, \dots$. Moreover, if $s_0 \neq -1/2, -3/4, -5/8$, all these orbits are generic.

Proof. I) If f_0 belongs to $H_3^{2,1}$ (where $\lambda = -\gamma^{-1} < 0, c > 0, d < 0, bc = -1$) or $H_3^{3,1}$ (where $\lambda = -\gamma^{-1} < 0, c > 0, d > 0, bc = -1$), the bifurcation curves $\tilde{B}_{2m}^+, \tilde{B}_k^-$ on the parameter plane (μ, α) are given by formulae (64) with $k = 2m$, where s_0 satisfies (57). Then it follows from these formulae that all the curves $\tilde{B}_{2m}^+, \tilde{B}_{2m}^-$ mutually intersect and they intersect the axis $\alpha = 0$ at the points $\mu = -(d)^{-1}(s_0 - 1 + \dots)\lambda^{4m}$ and $\mu = -(d)^{-1}(s_0 + 3 + \dots)\lambda^{4m}$, and the axis $\mu = 0$ at the points $\alpha = -(dy^-)^{-1}(s_0 - 1 + \dots)\lambda^{2m}$ and $\alpha = -(dy^-)^{-1}(s_0 + 3 + \dots)\lambda^{2m}$. Then, if $-3 < s_0 < 1$, all the domains E_{2m}^2 with sufficiently large k contain the origin ($\mu = 0, \alpha = 0$).

Moreover, by the Rescaling Lemma 6, all the first return maps T_k have “the same expression” for $\mu = 0$ and $\alpha = 0$. Indeed, we obtain from (33) that the rescaled form (32) of T_k takes the form (69) with $k = 2m$. Then, if $-3 < s_0 < 1$ every map (70) with sufficiently large $k = 2m$ has an elliptic fixed point which is generic if $s_0 \neq 0, -5/4$, see Section 4.2.1.

⁹Recall that $\alpha = cx^+/y^- - 1$ and $\tilde{\alpha} = cx^+/y^- + 1$, i.e., $\tilde{\alpha} = \alpha + 2$ and both $\alpha = 0$ and $\tilde{\alpha} = 0$ correspond to $\tau = 0$, see (23).

Since $c > 0$ in the case under consideration, it follows from (23) that $\alpha > -1$. This means that the curves $\tilde{B}_{2m+1}^{\pm 1}$ and \tilde{B}_{2m+1}^{2-} , given by formulae (66) with $k = 2m + 1$, as well as the corresponding domains \tilde{E}_{2m+1}^2 do not mutually intersect for different sufficiently large m . Moreover, they accumulate to the axis $\mu = 0$ as $m \rightarrow \infty$ from one side ($\mu > 0$, since $\alpha + 2 > 0$ and $\lambda^k < 0$ for odd k). This completes the proof for the case I.

II. Let now f_0 belong to $H_3^{2,2}$ (where $\lambda = -\gamma^{-1} < 0, c < 0, d < 0, bc = -1$) or $H_3^{3,2}$ (where $\lambda = -\gamma^{-1} < 0, c < 0, d > 0, bc = -1$). Therefore, since $\alpha < -1$, in contrast to the previous case, the bifurcation curves \tilde{B}_{2m}^+ and \tilde{B}_{2m}^- (see formula (64) with $k = 2m$) as well as the corresponding domains \tilde{E}_{2m} do not mutually intersect in \tilde{V} . Moreover, they accumulate to the axis $\mu = 0$ as $m \rightarrow \infty$ from one side ($\mu > 0$, since $\alpha < -1$ and $\lambda^k > 0$ for even k).

Since $\alpha < -1$ in the case under consideration, the curves $\tilde{B}_{2m+1}^{\pm 1}$ and \tilde{B}_{2m+1}^{2-} , given by formulae (66) with $k = 2m + 1$, can now intersect in \tilde{V} for different sufficiently large m . In this case the intersection points with the axis $\tilde{\alpha} = 0$ have the coordinates $\mu = -(d)^{-1}(s_0^{nor} + \dots)\lambda^{2k}$ and $\mu = -(d)^{-1}(s_0 + 1 + \dots)\lambda^{2k}$ with $k = 2m + 1$; and with the axis $\mu = 0$ the coordinates $\tilde{\alpha} = -(dy^-)^{-1}(s_0^{nor} + \dots)\lambda^{2m+1}$ and $\tilde{\alpha} = -(dy^-)^{-1}(s_0^{nor} + 1 + \dots)\lambda^{2m+1}$. Then if $-1 < s_0 < 0$, all the domains \tilde{E}_{2m+1}^2 with sufficiently large m contains the origin ($\mu = 0, \tilde{\alpha} = 0$) of \tilde{V} .

Moreover, by the Rescaling Lemma 6, all the first return maps T_{2m+1} have “the same expression” for $\mu = 0$ and $\tilde{\alpha} = 0$ (i.e., $cx^+ = -y^-$). Indeed, we obtain from (34) that the rescaled form (32) of T_k takes the form

$$\bar{X} = Y + O(k\lambda^{2k}), \bar{Y} = (-s_0^{nor} + \rho_k^4) + X - \frac{f_{03}}{d^2}\lambda^k Y^3 + O(k\lambda^{2k}), \quad (71)$$

where $k = 2m + 1$ and $\rho_k^4 = O(k\lambda^k)$ is a small coefficient (a correction to s_0^{nor}). (This map differs from (71) only by the sign in front of Y^3). Then, if $-1 < s_0^{nor} < 0$ every map (71) with sufficiently large $k = 2m + 1$ has an elliptic 2-periodic orbit which is generic, if $s_0 \neq -1/2, -3/4, -5/8$, see Section 4.2.2. \square

In Figures 17 and 18 we give an illustration of this theorem for different cases.

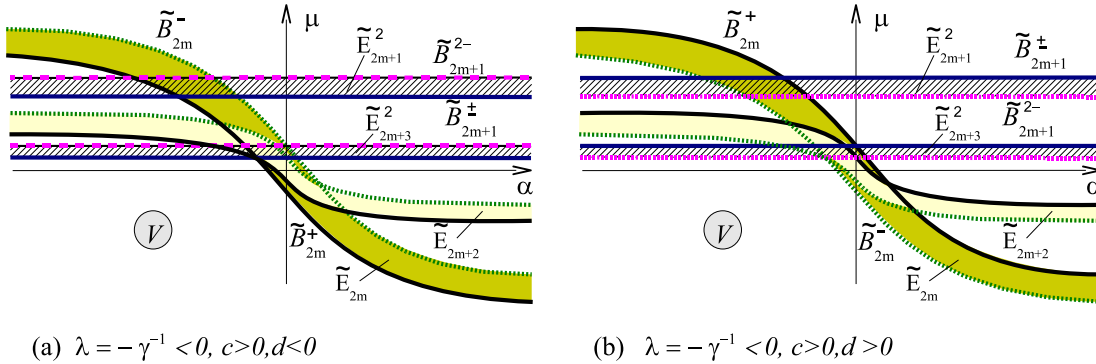


Figure 17: Elements of the bifurcation diagram in a neighbourhood $V(\mu = 0, \alpha = 0)$ for the families $f_{\mu, \alpha}$ in the cases where (a) $f_0 \in H_3^{2,1}$; (b) $f_0 \in H_3^{3,1}$.

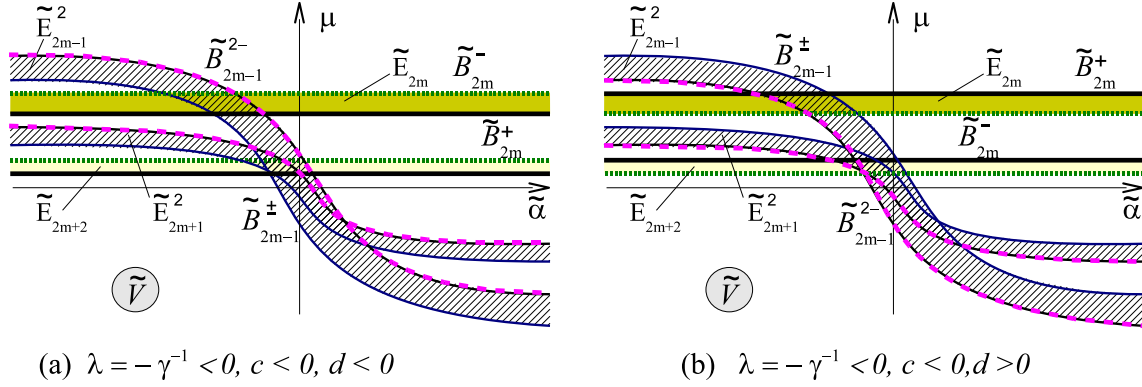


Figure 18: Elements of the bifurcation diagram in a neighbourhood $\tilde{V}(\mu = 0, \tilde{\alpha} = 0)$ for the families $f_{\mu, \alpha}$ in the cases where (a) $f_0 \in H_3^{2,2}$; (b) $f_0 \in H_3^{3,2}$.

7 Invariants of homoclinic tangencies.

We have just seen that in the case of global resonance $\tau = 0$ the dynamics of APMs of the third class (except for maps in H_3^5) depends, indeed, only on the quantity s_0 . In this section we prove, for completeness, the invariance of s_0 .

First, we recall the result from [43] that the quantity τ is an invariant of two-dimensional diffeomorphisms with homoclinic tangencies to a saddle with $\sigma \equiv |\lambda\gamma| = 1$. In particular, it was proved in [43] that the value of τ does not depend neither on the choice of pairs of homoclinic points M^+ and M^- nor in the coordinate changes conserving the first order normal form of the saddle map T_0 . This implies that, in the case of APMs, τ is invariant in those C^r -coordinates which conserve the first order normal form (3) of the saddle map T_0 . Note that, as it was shown in [57], τ is also invariant under C^1 -linearization coordinates. In principle, this result could be used for proving the existence/absence of topological Smale horseshoes near a homoclinic tangency.

We now prove the invariance of s_0 . However, in contrast to τ , we prove the invariance of s_0 in those C^{r-2} -coordinates which conserve the second order normal form (10) of the local map T_0 (or any n -order normal form (4) for $n \geq 2$). Naturally, s_0 “disappears” when a C^1 -linearization is used, since s_0 depends on the coefficients of T_1 in the quadratic terms which become indefinite for C^1 -changes.

Lemma 7. *Let $f_0 \in H_3^1 \cup H_3^4$ and $\tau = 0$. Then, in those coordinates where the local map T_0 takes the second normal form (10), the value of s_0 does not depend on the choice of pairs of homoclinic points of the orbit Γ_0 .*

Proof. We take first the pair $M^{+'} = T_0(M^+)$ and M^- of points of Γ_0 . Then the new global map $T_1' = T_0 T_1 : \Pi^- \rightarrow T_0(\Pi^+)$ can be written, by (10), in the form

$$\bar{x}' = \lambda \bar{x}(1 + \beta_1 \bar{x} \bar{y}) + O[\bar{x}^3 \bar{y}^2], \quad \bar{y}' = \gamma \bar{y}(1 - \beta_1 \bar{x} \bar{y}) + O[\bar{x}^2 \bar{y}^3], \quad (72)$$

where $\bar{x} = x^+ + F(x, y - y^-)$, $\bar{y} = G(x, y - y^-)$. We will calculate the corresponding coefficients (that define a new s_0') at the homoclinic point $M^-(x = 0, y = y^-)$ using that $\bar{x} = x^+, \bar{y} = 0$, $G_y(0, 0) = 0$ at this point. It follows from (72) that

$$\frac{\partial \bar{y}'}{\partial \bar{x}} = 0, \quad \frac{\partial \bar{x}'}{\partial \bar{x}} = \lambda, \quad \frac{\partial \bar{y}'}{\partial \bar{y}} = \gamma \quad \text{at } \bar{x} = x^+, \bar{y} = 0$$

and the O -terms in (72) vanish for $\bar{y} = 0$ along with all the required derivatives (note that only the second derivatives of \bar{y}' are needed). Thus, we have

$$\begin{aligned} a' &= \frac{\partial \bar{x}'}{\partial x} = \lambda \frac{\partial F}{\partial x} + \lambda \beta_1 (\bar{x})^2 \frac{\partial G}{\partial x} + O(\bar{y}), \quad c' = \frac{\partial \bar{y}'}{\partial x} = \gamma \frac{\partial G}{\partial x} + O(\bar{y}), \\ d' &= \frac{1}{2} \frac{\partial^2 \bar{y}'}{\partial y^2} = \frac{1}{2} \gamma \frac{\partial^2 G}{\partial y^2} + O(\bar{y}) + O(\partial \bar{y} / \partial y), \\ f'_{20} &= \frac{1}{2} \frac{\partial^2 \bar{y}'}{\partial x^2} = \frac{1}{2} \gamma \left(\frac{\partial^2 G}{\partial x^2} - 2\beta_1 \bar{x} \left(\frac{\partial G}{\partial x} \right)^2 \right) + O(\bar{y}), \\ f'_{11} &= \frac{\partial^2 \bar{y}'}{\partial x \partial y} = \gamma \frac{\partial^2 G}{\partial x \partial y} + O(\bar{y}) + O(\partial \bar{y} / \partial y). \end{aligned}$$

Since we calculate these derivatives at the point $x = 0, y = y^-$, where $\bar{x} = x^+, \bar{y} = 0$ and $\partial \bar{y} / \partial y \equiv G_y = 0$, we get

$$x^{+'} = \lambda x^+, \quad a' = \lambda a + \lambda (x^+)^2 \beta_1 c, \quad c' = \gamma c, \quad d' = \gamma d, \quad f'_{20} = \gamma f_{20} - \gamma c^2 \beta_1 x^+, \quad f'_{11} = \gamma f_{11}. \quad (73)$$

Then, by (35), we obtain that

$$\begin{aligned} s'_0 &= d' x^{+'} (a' c' + f'_{20} x^{+'}) + \frac{1}{2} f'_{11} x^{+'} (1 + \nu_1 - \frac{1}{2} f'_{11} x^{+'}) = \\ &= \lambda \gamma d x^+ ((a + (x^+)^2 \beta_1 c) c + (f_{20} - c^2 \beta_1 x^+) x^+) + \lambda \gamma \frac{1}{2} f_{11} x^+ (1 + \nu_1 - \frac{1}{2} f_{11} x^+) = s_0. \end{aligned}$$

We take now the pair $M^{+'} = M^+$ and $M^{-'} = T_0^{-1}(M^-)$ of points of Γ_0 . Then the new global map $T'_1 = T_1 T_0 : T_0^{-1}(\Pi^-) \rightarrow \Pi^+$ can be written as

$$\bar{x} = x' + F(x', y' - y^-), \quad \bar{y} = G(x', y' - y^-),$$

where $x' = \lambda x(1 + \beta_1 xy) + O(x^3 y^2)$ and $y' = \gamma y(1 - \beta_1 xy) + O(x^2 y^3)$ are coordinates in Π^- and $(x, y) \in T_0^{-1}\Pi^-$, $(\bar{x}, \bar{y}) \in \Pi^+$. Thus, we have that $x^{+'} = x^+, y^{-'} = \gamma^{-1} y^-$. Further, we calculate other coefficients as the corresponding derivatives of (\bar{x}, \bar{y}) with respect to (x, y) calculated at the point $x = 0, y = \gamma^{-1} y^-$. We get

$$\begin{aligned} a' &= \frac{\partial F}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial F}{\partial y'} \frac{\partial y'}{\partial x}, \quad c' = \frac{\partial G}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial G}{\partial y'} \frac{\partial y'}{\partial x}, \\ f'_{11} &= \frac{\partial^2 G}{(\partial x')^2} \frac{\partial x'}{\partial x} \frac{\partial x'}{\partial y} + \frac{\partial^2 G}{\partial x' \partial y'} \left(\frac{\partial y'}{\partial x} \frac{\partial x'}{\partial y} + \frac{\partial x'}{\partial x} \frac{\partial y'}{\partial y} \right) + \frac{\partial^2 G}{\partial y'^2} \frac{\partial y'}{\partial x} \frac{\partial y'}{\partial y} + \frac{\partial G}{\partial x'} \frac{\partial^2 x'}{\partial x \partial y} + \frac{\partial G}{\partial y'} \frac{\partial^2 y'}{\partial x \partial y}, \\ d' &= \frac{1}{2} \left(\frac{\partial^2 G}{(\partial x')^2} \left(\frac{\partial x'}{\partial y} \right)^2 + 2 \frac{\partial^2 G}{\partial x' \partial y'} \frac{\partial y'}{\partial y} \frac{\partial x'}{\partial y} + \frac{\partial^2 G}{\partial y'^2} \left(\frac{\partial y'}{\partial y} \right)^2 + \frac{\partial G}{\partial x'} \frac{\partial^2 x'}{\partial y^2} + \frac{\partial G}{\partial y'} \frac{\partial^2 y'}{\partial y^2} \right), \\ f'_{20} &= \frac{1}{2} \left(\frac{\partial^2 G}{(\partial x')^2} \left(\frac{\partial x'}{\partial x} \right)^2 + 2 \frac{\partial^2 G}{\partial x' \partial y'} \frac{\partial y'}{\partial x} \frac{\partial x'}{\partial x} + \frac{\partial^2 G}{\partial y'^2} \left(\frac{\partial y'}{\partial x} \right)^2 + \frac{\partial G}{\partial x'} \frac{\partial^2 x'}{\partial x^2} + \frac{\partial G}{\partial y'} \frac{\partial^2 y'}{\partial x^2} \right). \end{aligned}$$

Since

$$\frac{\partial G}{\partial y'} = 0, \quad \frac{\partial}{\partial y} \left(x', \frac{\partial x'}{\partial x}, \frac{\partial x'}{\partial y} \right) = 0, \quad \frac{\partial x'}{\partial x} = \lambda, \quad \frac{\partial y'}{\partial y} = \gamma, \quad \frac{\partial y'}{\partial x} = -\gamma^{-1} \beta_1 (y^-)^2$$

at the point $M^{-'} (x = 0, y = \gamma^{-1}y^{-})$, we obtain that

$$\begin{aligned} a' &= \lambda a - b\beta_1\gamma^{-1}(y^{-})^2, \quad c' = \lambda c, \quad f'_{11} = \lambda\gamma f_{11} - 2d\beta_1(y^{-})^2, \quad d' = d\gamma^2, \\ f'_{20} &= f_{20}\lambda^2 - f_{11}\lambda^2\beta_1(y^{-})^2 + d\lambda^2\beta_1^2(y^{-})^4 + c\lambda^2\beta_1y^{-}, \end{aligned} \quad (74)$$

Since $\lambda\gamma = 1$, we obtain, by (35), that

$$\begin{aligned} s'_0 &= d'x^{+'}(a'c' + f'_{20}x^{+'}) + \frac{1}{2}f'_{11}x^{+'}(1 + \nu_1 - \frac{1}{2}f'_{11}x^{+'}) = \\ &= dx^+[ac - cb\beta_1(y^{-})^2 + f_{20}x^+ - f_{11}x^+\beta_1(y^{-})^2 + dx^+\beta_1^2(y^{-})^4 + c\beta_1x^+y^-] + \\ &+ \frac{1}{2}(f_{11}x^+ - 2d\beta_1(y^{-})^2x^+)(1 + \nu_1 - \frac{1}{2}(f_{11}x^+ - 2d\beta_1(y^{-})^2x^+)) = \\ &= s_0 + d\beta_1x^+y^-(cx^+ - bcy^- - y^-(1 + \nu_1)). \end{aligned}$$

Note that $\nu_1 = -bc$ in the case $\lambda\gamma = 1$ and, thus, $s'_0 = s_0 + d\beta_1x^+y^-(cx^+ - y^-)$. It follows that $s'_0 = s_0$ at $cx^+ = y^-$ which is equivalent to $\tau = 0$ if $c > 0$. \square

In the locally non-orientable case, s_0 remains invariant with respect to the choice of any pair of homoclinic points of the needed type (see the definition of homoclinic points of the needed time just before the condition **D** in Section 2.2).

Lemma 8. *Let $f_0 \in H_3^2 \cup H_3^3$. Then, in coordinates where the map T_0 takes the second normal form (10), the value of s_0 does not depend on the choice of pairs of homoclinic points of the needed type.*

Proof. By condition **D**, the pair M^+ and M^- of homoclinic points is of the needed type (i.e., the corresponding map T_1 is orientable). We prove the invariance of s_0 for the pairs a) $T_0^2(M^+)$ and M^- ; b) M^+ and $T_0^{-2}(M^-)$ and c) $T_0(M^+)$ and $T_0^{-1}(M^-)$, which are all of the needed type. Note that in the case $\lambda\gamma = -1$ the calculations become much simpler, since $\beta_1 = 0$.

a) For the pair $M^{+'} = T_0(M^+)$ and $M^{-'} = M^-$ of homoclinic points, we obtain from (73) that

$$x^{+'} = \lambda x^+, \quad a' = \lambda a, \quad c' = \gamma c, \quad d' = \gamma d, \quad f'_{20} = \gamma f_{20}, \quad f'_{11} = \gamma f_{11}. \quad (75)$$

Since $\lambda\gamma = -1$, we obtain then that $s'_0 = -s_0$. Analogously, for the pairs $M^{+''} = T_0(M^{+'})$ and $M^{-''} = M^-$, we obtain that $s''_0 = -s'_0$ and, hence $s''_0 = s_0$.

b) For the pair $M^{+'} = M^+$ and $M^{-'} = T_0^{-1}(M^-)$ of homoclinic points, we have that $x^{+'} = x^+$ and, by (74),

$$a' = \lambda a, \quad c' = \lambda c, \quad f'_{11} = -f_{11}, \quad d' = d\gamma^2, \quad f'_{20} = f_{20}\lambda^2. \quad (76)$$

Since $\lambda\gamma = -1$, we obtain by (35) that

$$[dx^+(ac + f_{20}x^+)]' = dx^+(ac + f_{20}x^+) \quad \text{and} \quad [f_{11}x^+]' = -f_{11}x^+. \quad (77)$$

This implies evidently that $s''_0 = s_0$ for the needed type pair $M^{+''} = M^+$ and $M^{-''} = T_0^{-2}(M^-)$ of homoclinic points.

c) Consider first the pair $M^{+'} = T_0(M^+)$ and $M^{-'} = M^-$ for which formula (75) holds with a negative coordinate $x^{+'}$ of the point $M^{+'}$. Therefore, we make the coordinate change $x \rightarrow -x, y \rightarrow y$ after which the new map T'_1 will have the following coefficients

$$x^{+'} = -\lambda x^+, \quad a' = \lambda a, \quad c' = -\gamma c, \quad f'_{11} = -\gamma f_{11}, \quad d' = d\gamma, \quad f'_{20} = f_{20}\gamma,$$

which gives relation (77). Evidently, at the further transition to the pair $M^{+''} = T_0(M^+)$ and $M^{-''} = T_0^{-1}(M^-)$, this gives the required equality $s''_0 = s_0$. \square

8 The proof of Lemma 2.

We start from the well-known fact that the local stable and unstable manifolds of O can be straightened out by means of a certain C^r -symplectic change of coordinates,¹⁰ i.e., the map T_0 can be written in the following form

$$\bar{x} = \lambda(\varepsilon)x + f(x, y, \varepsilon)x, \quad \bar{y} = \gamma(\varepsilon)y + g(x, y, \varepsilon)y, \quad (78)$$

where $f(0, 0, \varepsilon) \equiv 0, g(0, 0, \varepsilon) \equiv 0$. In these coordinates, the fixed point O_ε is in the origin and the equations of W_{loc}^s and W_{loc}^u are $y = 0$ and $x = 0$, respectively, for all sufficiently small ε .

We consider the map T_ε in the initial form (78). This map is C^r and can be represented in the following “ n -th order extended form”

$$\begin{aligned} \bar{x} = & \lambda(\varepsilon)x\{1 + [\varphi_1^{(0)}(x, \varepsilon) + \psi_1^{(0)}(y, \varepsilon)] + [\beta_1^{(1)} + \varphi_1^{(1)}(x, \varepsilon) + \psi_1^{(1)}(y, \varepsilon)] \cdot xy + \\ & + [\beta_1^{(2)} + \varphi_1^{(2)}(x, \varepsilon) + \psi_1^{(2)}(y, \varepsilon)] \cdot (xy)^2 + \dots + \\ & + [\beta_1^{(n)} + \varphi_1^{(n)}(x, \varepsilon) + \psi_1^{(n)}(y, \varepsilon)] \cdot (xy)^n\} + O(x^{n+2}y^{n+1}), \\ \bar{y} = & \gamma(\varepsilon)y\{1 + [\varphi_2^{(0)}(x, \varepsilon) + \psi_2^{(0)}(y, \varepsilon)] + [\beta_2^{(1)} + \varphi_2^{(1)}(x, \varepsilon) + \psi_2^{(1)}(y, \varepsilon)] \cdot xy \\ & + [\beta_2^{(2)} + \varphi_2^{(2)}(x, \varepsilon) + \psi_2^{(2)}(y, \varepsilon)] \cdot (xy)^2 + \dots + \\ & + [\beta_2^{(n)} + \varphi_2^{(n)}(x, \varepsilon) + \psi_2^{(n)}(y, \varepsilon)] \cdot (xy)^n\} + O(x^{n+1}y^{n+2}) \end{aligned} \quad (79)$$

where $|\lambda\gamma| = 1$, $\beta_1^{(i)}$ and $\beta_2^{(i)}$ are constants, $i = 1, \dots, n$, $\varphi_k^{(i)}(0, \varepsilon) = \psi_k^{(i)}(0, \varepsilon) \equiv 0$, $k = 1, 2$. Denote $\alpha_{ki} \equiv [\varphi_k^{(i)}(x, \varepsilon) + \psi_k^{(i)}(y, \varepsilon)]$. Since $T_\varepsilon \in C^r$, we have, due to the expansion in (79), that $\alpha_{ki} \in C^{r-2i-1}$.

Lemma 2 states that there exist canonical changes which cancel the functions α_{ki} and transform constants β_1^i and β_2^i into the “Birkhoff-Moser coefficients” $\tilde{\beta}_i$ and $\tilde{\beta}_i$, respectively. In making these changes we will see that the change cancelling the term α_{ki} is C^{r-2i-2} , while the next term $\alpha_{k,i+1}$ is $C^{r-2(i+1)-1} = C^{r-2i-3}$. That is, such a change will not change the smoothness of the high order terms (in the sense of the expansion in (79)). Thus, the final smoothness will be equal to the smoothness of the last coordinate transformation.

Now we prove the lemma by induction on i . Note that Lemma 1 can be considered here as “the first step of induction”.

Suppose that for some $i \leq n$ we have brought the map T_ε to the form

$$\begin{aligned} \bar{x} = & \lambda(\varepsilon)x\{1 + \beta_1(\varepsilon) \cdot xy + \beta_2(\varepsilon) \cdot (xy)^2 + \dots + \beta_{i-1}(\varepsilon) \cdot (xy)^{i-1} + \\ & + \beta_1^{(i)} + [\varphi_1^{(i)}(x, \varepsilon) + \psi_1^{(i)}(y, \varepsilon)] \cdot (xy)^i\} + O(x^{i+2}y^{i+1}), \\ \bar{y} = & \gamma(\varepsilon)y\{1 + \tilde{\beta}_1(\varepsilon) \cdot xy + \tilde{\beta}_2(\varepsilon) \cdot (xy)^2 + \dots + \tilde{\beta}_{i-1}(\varepsilon) \cdot (xy)^{i-1} + \\ & + \tilde{\beta}_2^{(i)} + [\varphi_2^{(i)}(x, \varepsilon) + \psi_2^{(i)}(y, \varepsilon)] \cdot (xy)^i\} + O(x^{i+1}y^{i+2}) \end{aligned} \quad (80)$$

Let us show that there exists a canonical change cancelling the terms α_{1i} and α_{2i} and that the smoothness of such a change is equal to the smoothness of functions $\alpha_{k,i}$ minus one. Then, the lemma will be proven.

For this goal we make two consecutive canonical changes with the following generating functions

$$V_1^{(i)}(x, \eta) = x\eta + (x\eta)^{i+1}v_1^{(i)}(x, \varepsilon) \quad \text{and} \quad V_2^{(i)}(x, \eta) = x\eta + (x\eta)^{i+1}v_2^{(i)}(\eta, \varepsilon), \quad (81)$$

¹⁰Let us recall some details of this. We can always write the local map in the form $\bar{x} = \lambda(\varepsilon)x + h_1(x, y, \varepsilon)$, $\bar{y} = \gamma(\varepsilon)y + h_2(x, y, \varepsilon)$, where $|\lambda\gamma| = 1$, $h_i(0, 0, \varepsilon) = 0$. Let $y = \varphi(x, \varepsilon)$ be the equation of W_{loc}^s . Then, by the change $\xi = x, \eta = y - \varphi(x, \varepsilon)$, we straighten out W_{loc}^s . Moreover, this change is symplectic, since it is produced by the generating function $V(x, \eta, \varepsilon) = x\eta + \int \varphi(x, \varepsilon)dx$. The manifold W_{loc}^u is straightened out analogously.

where $v_k^{(i)}(0, \varepsilon) = 0$, $k = 1, 2$. By means of these changes one can vanish functions $\varphi_1^{(i)}$ and $\psi_2^{(i)}$ in (80), respectively. After this, we show that the new functions $\tilde{\varphi}_2^{(i)}$ and $\tilde{\psi}_1^{(i)}$ vanish due to equality to one of $|J(T_\varepsilon)|$.

First, we make the change associated to the generating function $V_1^{(i)}$ where $v_1^{(i)}(0, \varepsilon) = 0$. Thus, this change is

$$\xi = x + (i+1)x^{i+1}\eta^i v_1^{(i)}(x, \varepsilon), \quad y = \eta + x^i \eta^{i+1} \tilde{v}_1^{(i)}(x, \varepsilon) \quad (82)$$

where $\tilde{v}_1^{(i)}(x, \varepsilon) = (i+1)v_1^{(i)}(x, \varepsilon) + x \cdot \partial v_1^{(i)} / \partial x$ and $\tilde{v}_1^{(i)}(0, \varepsilon) \equiv 0$.

The first equation of (80) is transformed to

$$\begin{aligned} \bar{\xi} &= \bar{x} + (i+1)\bar{x}^{i+1}\bar{\eta}^i \bar{v}_1^{(i)}(\bar{x}, \varepsilon) = \lambda x \{1 + \beta_1 \cdot xy + \beta_2 \cdot (xy)^2 + \dots \\ &\quad + \beta_{i-1} \cdot (xy)^{i-1} + \beta_1^{(i)} \cdot (xy)^i + \varphi_1^{(i)}(x, \varepsilon) \cdot (xy)^i + \\ &\quad + \psi_1^{(i)}(y, \varepsilon) \cdot (xy)^i\} + (i+1)\lambda^{i+1}x^{i+1}\gamma^i y^i v_1^{(i)}(\lambda x, \varepsilon) + O(\xi^{i+2}\eta^{i+1}) \\ &= \lambda \xi + x^{i+1}y^i \left[-(i+1)\lambda v_1^{(i)}(x, \varepsilon) + (i+1)\lambda \delta_i v_1^{(i)}(\lambda x, \varepsilon) + \lambda \varphi_1^{(i)}(x, \varepsilon) \right] + \\ &\quad + \lambda \xi \{ \beta_1 \cdot \xi \eta + \beta_2 \cdot (\xi \eta)^2 + \dots + \beta_{i-1} \cdot (\xi \eta)^{i-1} + \beta_1^{(i)} \cdot (\xi \eta)^i + \\ &\quad + \psi_1^{(i)}(\eta, \varepsilon) \cdot \xi (\xi \eta)^i \} + O(\xi^{i+2}\eta^{i+1}), \end{aligned} \quad (83)$$

where $\delta_i = \text{sign}(\lambda \gamma)^i$. Now we take a function $v_1^{(i)}(x, \varepsilon)$ to cancel the expression inside the square brackets in (83), i.e.,

$$v_1^{(i)}(\lambda x, \varepsilon) = \delta_i v_1^{(i)}(x, \varepsilon) - \frac{1}{i+1} \varphi_1^{(i)}(x, \varepsilon) \quad (84)$$

Note that this equation has a solution in the class of functions (of variable x) whose smoothness coincides with the smoothness of the function $\varphi_1^{(i)}(x, \varepsilon)$ (recall that $\varphi_1^{(i)} \in C^{r-2i-1}$). The sought solution, $u = v_1^{(i)}(x, \varepsilon)$, can be viewed as the equation of the strong stable invariant manifold W_i^{ss} containing the point $(0, 0)$ of the following planar map

$$\bar{u} = \delta_i u - \frac{1}{i+1} \varphi_1^{(i)}(x, \varepsilon), \quad \bar{x} = \lambda(\varepsilon)x \quad (85)$$

(since W^{ss} is invariant, its equation $u = \phi_{ss}(x, \varepsilon)$ has to satisfy the following homological equation: $\phi_{ss}(\lambda x, \varepsilon) = \delta_i \phi_{ss}(x, \varepsilon) - \frac{1}{i+1} \varphi_1^{(i)}(x, \varepsilon)$ that is, (84). Since $\delta_i = \pm 1$, such a manifold exists, it is C^{r-2i-1} and, thus, the change (84) is C^{r-2i-2}).

We can see from (82) that the sought change is of the form

$$x = \xi + O((\xi \eta)^{i+1}), \quad y = \eta + O((\xi \eta)^{i+1}).$$

This means that, in the second equation of (80), such a change can affect only the function $\lambda^{-1} \varphi_2^{(i)}(x, \varepsilon) x^i y^{i+1}$ from the explicitly shown ones in (80): $\varphi_2^{(i)} \Rightarrow \tilde{\varphi}_2^{(i)}$.

Thus, after change (82), the map T_ε has the form (80) where

$$\varphi_1^{(i)}(x, \varepsilon) \equiv 0, \quad \varphi_2^{(i)} \equiv \tilde{\varphi}_2^{(i)}, \quad (86)$$

and the other explicitly given functions are the same. Note that the function $\psi_2^{(i)}(y, \varepsilon)$ does not change.

It is evident that the second coordinate transformation, associated to the second generating function $V_2^{(i)} = x\eta + (x\eta)^{i+1}v_2^{(i)}(\eta, \varepsilon)$ with $v_2^{(i)}(0, \varepsilon) = 0$, is carried out quite similarly, due to the condition $|\lambda \gamma| \equiv 1$, see also [5].

Thus, after the canonical changes with associated generating functions $V_1^{(i)}$ and $V_2^{(i)}$ from (81), the map T_ε takes the following form

$$\begin{aligned}\bar{x} &= \lambda(\varepsilon)x\{1 + \beta_1(\varepsilon) \cdot xy + \dots + \beta_i(\varepsilon) \cdot (xy)^i\} + \tilde{\psi}_1^{(i)}(y, \varepsilon) \cdot x^{i+1}y^i + O(x^{i+2}y^{i+1}), \\ \bar{y} &= \gamma(\varepsilon)y\{1 + \tilde{\beta}_1(\varepsilon) \cdot xy + \dots + \tilde{\beta}_i(\varepsilon) \cdot (xy)^i\} + \tilde{\varphi}_2^{(i)}(x, \varepsilon) \cdot x^iy^{i+1} + O(x^{i+1}y^{i+2})\end{aligned}\quad (87)$$

Let us show that the equality $J(T_\varepsilon) \equiv 1$ implies $\tilde{\psi}_1^{(i)} \equiv 0$ and $\tilde{\varphi}_2^{(i)} \equiv 0$. Indeed, we can represent the map (87) as

$$\begin{aligned}\bar{x} &= \lambda(\varepsilon)x B_i(xy) + \tilde{\psi}_1^{(i)}(y, \varepsilon) \cdot x^{i+1}y^i + O(x^{i+2}y^{i+1}), \\ \bar{y} &= \gamma(\varepsilon)y B_i^{-1}(xy) \tilde{\varphi}_2^{(i)}(x, \varepsilon) \cdot x^iy^{i+1} + O(x^{i+1}y^{i+2})\end{aligned}\quad (88)$$

where B_i and B_i^{-1} are the truncations of the Birkhoff-Moser normal form. Then the Jacobian of (88) has the following form

$$J = \pm 1 + (i+1)(\lambda \tilde{\varphi}_2^{(i)}(x, \varepsilon) + \gamma \tilde{\psi}_1^{(i)}(y, \varepsilon)) \cdot x^iy^i + O((xy)^{i+1}),$$

from which it follows that $\tilde{\varphi}_2^{(i)} \equiv 0$ and $\tilde{\psi}_1^{(i)} \equiv 0$.

In the non-orientable case $\lambda\gamma = -1$, the monomials of the form $\beta_i x(xy)^i$ in the equation for \bar{x} and $\tilde{\beta}_i y(xy)^i$ in the equation for \bar{y} with odd i are non-resonant. Therefore, they can be cancelled (inside every corresponding step of the proof) by the canonical polynomial coordinate transformations with generating functions $\tilde{V}_i = x\eta + \nu_i(x\eta)^{i+1}$. One can check that if in (80) all terms β_i and $\tilde{\beta}_i$ vanish for odd i , except for the last ones β_n and $\tilde{\beta}_n$ for odd n , then $\beta_n = -\tilde{\beta}_n$. Then the change with the generating functions \tilde{V}_n cancels both these terms simultaneously.

This completes the proof of the lemma.

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References

- [1] Mora L., Romero N. Moser’s invariant curves and homoclinic bifurcations.- Dyn. Sys. and Appl., 1997, v. 6, 29-41.
- [2] Gonchenko S.V., Shilnikov L.P. On two-dimensional area-preserving mappings with homoclinic tangencies.- Doklady Mathematics, 2001, v. 63, no. 3, 395-399.
- [3] Gonchenko S.V., Shilnikov L.P. On two-dimensional area-preserving maps with homoclinic tangencies that have infinitely many generic elliptic periodic points.- Notes of Saint-Petersburg Steklov Math.Inst., 2003, v. 300, 155-166.

- [4] Duarte, P. Elliptic isles in families of area-preserving maps, *Ergodic Theory Dynam. Systems*, 2008, v. 28, no. 6, 1781-1813.
- [5] Gonchenko S.V., Gonchenko M.S. On cascades of elliptic periodic points in two-dimensional symplectic maps with homoclinic tangencies.- *J. Regular and Chaotic Dynamics*, 2009, v. 14, no. 1, 116-136.
- [6] Biragov V.S. Bifurcations in a two-parameter family of conservative mappings that are close to the Hénon map.- *Selecta Math.Sov.*, 1990, v. 9, 273-282. [Originally publ. in "Methods of qualitative theory of differential equations", Gorky State Univ., 1987, 10-24.]
- [7] Biragov V.S., Shilnikov L.P. On the bifurcation of a saddle-focus separatrix loop in a three-dimensional conservative system.- *Sel. Math. Sov.*, 1992, v. 11, 333-340. [Originally publ. in "Methods of qualitative theory and theory of bifurcations", Gorky St. Univ., 1989, 25-34.]
- [8] Lerman L.M. Complex dynamics and bifurcations in a Hamiltonian system having a transversal homoclinic orbit to a saddle focus.- *Chaos*, 1991, v. 1, no. 2, 174-180.
- [9] Lerman L.M. Dynamical Phenomena near a Saddle-Focus Homoclinic Connection in a Hamiltonian System.- *Journal of Statistical Physics*, 2000, v. 101, nos. 1-2, 357-372.
- [10] Lerman L., Koltsova O. Hamiltonian dynamics near nontransverse homoclinic orbit to saddle-focus equilibrium.- *Discrete and Continuous Dynamical Systems*, 2009, v. 25, no. 3, 883-913.
- [11] Gavrilov N.K., Shilnikov L.P. On three-dimensional dynamical systems close to systems with a structurally unstable homoclinic curve.- I, *Math. USSR Sbornik*, 1972, v. 17, 467-485.
- [12] Newhouse S.E. Diffeomorphisms with infinitely many sinks. *Topology*, 1974, v. 13, 9-18.
- [13] Gonchenko S.V. On stable periodic motions in systems close to a system with a nontransversal homoclinic curve. *Russian Math. Notes*, 1983, v. 33, no. 5, 384-389.
- [14] Palis J., Viana M. High dimension diffeomorphisms displaying infinitely many sinks. *Ann. Math.*, 1994, v. 140, 91-136.
- [15] Newhouse S.E. The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms. *Publ. Math. Inst. Hautes Etudes Sci.*, 1979, v. 50, 101-151.
- [16] Gonchenko S.V., Turaev D.V. and Shilnikov L.P. On the existence of Newhouse regions near systems with non-rough Poincare homoclinic curve (multidimensional case). *Russian Acad. Sci.Dokl.Math.*, 1993, v. 47, 268-273.
- [17] Romero N. Persistence of homoclinic tangencies in higher dimensions.- *Ergod. Th. and Dyn.Sys.*, 1995, v. 15, 735-757.
- [18] Duarte, P. Abundance of elliptic isles at conservative bifurcations. *Dynam. Stability Systems* 14, 1999, no. 4, 339-356.
- [19] Duarte P. Persistent homoclinic tangencies for conservative maps near the identity.- *Ergod. Th. and Dyn. Sys.*, 2000, v. 20, no. 2, 393-438.
- [20] Mora, L., Romero, N. Persistence of homoclinic tangencies for area-preserving maps. *Ann. Fac. Sci. Toulouse Math. (6)* 6, 1997, no. 4, 711-725.

- [21] Homoclinic Tangencies (S.V.Gonchenko, L.P.Shilnikov eds.), Moscow-Izhevsk, 2007.
- [22] Gonchenko S.V., Gonchenko V.S. On Andronov-Hopf bifurcations of two-dimensional diffeomorphisms with homoclinic tangencies.- WIAS-preprint no. 556, Berlin, 2000, 27p.
- [23] Gonchenko S.V., Gonchenko V.S. On bifurcations of birth of closed invariant curves in the case of two-dimensional diffeomorphisms with homoclinic tangencies.- Proc. of Math.Steklov Inst., Moscow, 2004, v. 244, 80-105.
- [24] Gonchenko V.S., Kuznetsov Yu.A., Meijer H.G.E. Generalized Hénon map and bifurcations of homoclinic tangencies. SIAM J. of Appl. Dyn. Sys. 2005, v. 4. no. 2. 407-436.
- [25] Gonchenko M.S. On the structure of 1:4 resonances in Hénon maps.- Int.Journal of Bifurcation and Chaos, 2005, v. 15, no. 11, 3653-3660.
- [26] Gonchenko S.V., Markichev A.S., Shatalin A.E. On stable periodic orbits of two-dimensional diffeomorphisms close to a diffeomorphism with a non-transversal heteroclinic cycle.- Rus. Diff. Eq., 2001, v. 37, 205-215.
- [27] Gonchenko S.V., Turaev D.V., Shilnikov L.P. On Newhouse regions of two-dimensional diffeomorphisms close to a diffeomorphism with a nontransversal heteroclinic cycle.- Proc. Steklov Inst. Math., 1997, v. 216, 70-118.
- [28] Gonchenko S.V., Shilnikov L.P., Stenkin O.V. On Newhouse regions with infinitely many stable and unstable invariant tori.- Proc. Int.Conf. "Progress in Nonlinear Science" dedicated to 100th Anniversary of A.A.Andronov, July 2-6; v,1 "Mathematical Problems of Nonlinear Dynamics", Nizhni Novgorod, 2002, 80-102.
- [29] Gonchenko S.V., Sten'kin O.V., Shilnikov L.P. On the existence of infinitely stable and unstable invariant tori for systems from Newhouse regions with heteroclinic tangencies.- Nonlinear Dynamics, 2006, v. 2, no. 1, 3-25 (in Russian).
- [30] Lamb J.S.W., Sten'kin O.V. Newhouse regions for reversible systems with infinitely many stable, unstable and elliptic periodic orbits.- Nonlinearity, 2004, v. 17, 1217-1244.
- [31] Delshams A., Gonchenko S.V., Gonchenko V.S., Lazaro J.T. and Sten'kin O.V. Abundance of attracting, repelling and elliptic periodic orbits in two-dimensional reversible maps. Nonlinearity, 2013, 26, 1-33.
- [32] Pikovsky A., Topaj D. Reversibility vs. synchronization in oscillator lattices, Physica D, 2002, v. 170, 118-130.
- [33] Gonchenko S.V., Gonchenko A.S., Kazakov A.O. Richness of chaotic dynamics in nonholonomic models of a Celtic stone.- Regular and Chaotic Dynamics, 2013, v. 15, no. 5, 521-538.
- [34] Kazakov A.O. Strange Attractors and Mixed Dynamics in the Problem of an Unbalanced Rubber Ball Rolling on a Plane.- Regular and Chaotic Dynamics, 2013, v. 18, no. 5, 508-520.
- [35] Newhouse S.E. Quasi-elliptic periodic points in conservative dynamical systems.- Amer. J. of Math., 1977, v. 99, 1061-1087.

- [36] Gonchenko S.V., Shilnikov L.P., Turaev D.V. Elliptic periodic orbits near a homoclinic tangency in four-dimensional symplectic maps and Hamiltonian systems with three degrees of freedom.- Regular and Chaotic Dynamics, 1998, v. 3, no. 4, 3-26.
- [37] Gonchenko S.V., Shilnikov L.P., Turaev D.V. Existence of infinitely many elliptic periodic orbits in four-dimensional symplectic maps with a homoclinic tangency.- Proc. Steklov Inst. Math., 2004, v. 244, 115-142.
- [38] Gonchenko S.V., Shilnikov L.P. On two-dimensional analytic area-preserving diffeomorphisms with infinitely many stable elliptic periodic points.- Regular and Chaotic Dynamics, 1997, v. 2, no. 3/4, 106-123.
- [39] Gonchenko S.V., Shilnikov L.P. On two-dimensional area-preserving diffeomorphisms with infinitely many elliptic islands.- J.of Stat.Phys., 2000, v. 101, no. 1/2, 321-356.
- [40] Duarte, P. Plenty of elliptic islands for the standard family of area preserving maps. Ann. Inst. H. Poincaré Anal. Non Linéaire, 1994, v. 11, no 4, 359-409.
- [41] De Simoi, J. On cyclicity-one elliptic islands of the standard map. J. Mod. Dyn., 2013, v. 7, no. 2, 153-208.
- [42] Moser J. The analytic invariants of an area-preserving mapping near a hyperbolic fixed point.- Comm. of Pure and Appl.Math., 1956, v. 9, 673-692.
- [43] Gonchenko S.V., Shilnikov L.P. Arithmetic properties of topological invariants of systems with a structurally unstable homoclinic trajectory.- Ukrainian Math.J., 1987, v. 39, no. 1, 21-28.
- [44] Gonchenko S.V., Shilnikov L.P. Invariants of Ω -conjugacy of diffeomorphisms with a structurally unstable homoclinic trajectory.- Ukrainian Math.J., 1990, v. 42, no. 2, 134-140.
- [45] Gonchenko S.V., Shilnikov L.P., Turaev D.V. Homoclinic tangencies of arbitrarily high orders in conservative and dissipative two-dimensional maps.- Nonlinearity, 2007, v. 20, 241-275.
- [46] Leontovich E.A. On a birth of limit cycles from a separatrix loop.- Soviet Math. Dokl., 1951, v. 78(4), 641-644.
- [47] Leontovich E.A. Birth of limit cycles from a separatrix loop of a saddle of a planar system in the case of zero saddle value.- Preprint, Moscow: VINITI, 1988.
- [48] Afraimovich V.S. On smooth changes of variables”.- Selecta Math.Sov., 1990, v. 9, no. 3, 205-214. [Originally publ. in “Methods of the Qualitative Theory and the Bifurcation Theory”, Gorky State Univ., 1984, 10-21].
- [49] Gonchenko S.V., Shilnikov L.P. On geometrical properties of two-dimensional diffeomorphisms with homoclinic tangencies.- Int.Journal of Bifurcation and Chaos, 1995, v. 5, no. 3, 819-829.
- [50] Gonchenko S.V., Shil’nikov L.P., Turaev D.V. On models with non-rough Poincare homoclinic curves.-Physica D, 1993, v. 62, nos. 1-4, 1-14.

- [51] Gonchenko S.V., Turaev D.V. and Shilnikov L.P. Homoclinic tangencies of an arbitrary order in Newhouse domains.- *Itogi Nauki Tekh., Ser. Sovrem. Mat. Prilozh., Temat. Obz.*, 1999, v. 67, 69-128 ; // English transl. in *J.Math.Sci.*, New York, 2001, v. 105, no. 1, 1738-1778.
- [52] Gonchenko S.V., Shilnikov L.P. On dynamical systems with structurally unstable homoclinic curves.- *Soviet Math. Dokl.*, 1986, v. 33, no. 1, 234-238.
- [53] Arnold V.I. *Geometrical Methods in the Theory of Ordinary Differential Equations*.- Springer; 2nd edition, 1996.
- [54] Arnold V.I., Afraimovich V.S., Ilyashenko Yu.S., and Shilnikov L.P. *Bifurcation Theory, Dynamical Systems V. Encyclopaedia of Mathematical Sciences*", Springer-Verlag, 1994.
- [55] Arnold V.I. Small denominators and problems of stability of motion in Classical and Celestial Mechanics.- *Russian Math. Surveys*, 1963, v. 18, no. 6.
- [56] Turaev D., Rom-Kedar V. Elliptic islands appearing in near-ergodic flows.- *Nonlinearity*, 1998, v. 11, 575-600.
- [57] Afraimovich V., Young T. Multipliers of homoclinic tangencies and a theorem of Gonchenko and Shilnikov on area preserving maps.- *Int. J. Bifurcation and Chaos*, 2005, v. 15, no. 11, 3589-3594.